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THE CHERN NUMBERS OF THE NORMALIZATION OF AN ALGEBRAIC THREEFOLD WITH ORDINARY SINGULARITIES

by

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Abstract. — By a classical formula due to Enriques, the Chern numbers of the non-singular normalization X of an algebraic surface S with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$ are given by $\int_X c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \gamma$, $\int_X c_2 = n(n^2-4n+6) - (3n-8)m + 3t - 2\gamma$, where $n =$ the degree of S, $m =$ the degree of the double curve (singular locus) D_S of S, $t =$ the cardinal number of the triple points of S, and γ=the cardinal number of the cuspidal points of S. In this article we shall give similar formulas for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ (Theorem 1.15, Theorem 2.1, Theorem 3.2). As a by-product, we obtain a numerical formula for the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X)$ with coefficient in the sheaf $\mathcal{T}_X$ of holomorphic vector fields on the non-singular normalization X of $\overline{X}$ (Theorem 4.1).

***Résumé* (Les nombres de Chern de la normalisée d'une variété algébrique de dimension 3 à points singuliers ordinaires)**

Par une formule classique due à Enriques, les nombres de Chern de la normalisation non singulière X de la surface algébrique S avec singularités ordinaires dans $\mathbb{P}^3(\mathbb{C})$ sont donnés par $\int_X c_1^2 = n(n-4)^2 - (3n-16)m + 3t - \gamma$, $\int_X c_2 = n(n^2-4n+6) - (3n-8)m + 3t - 2\gamma$, où n est le degré de S, m est le degré de la courbe double (lieu singulier) D_S de S, t est le nombre de points triples de S, et γ est le nombre de points cuspidaux de S. Dans cet article nous donnons des formules similaires pour une “threefold” algébrique $\overline{X}$ avec singularités ordinaires dans $\mathbb{P}^4(\mathbb{C})$ (Théorème 1.15, Théorème 2.1, Théorème 3.2). Comme application, nous obtenons une formule numérique pour la caractéristique d'Euler-Poincaré $\chi(X, \mathcal{T}_X)$ à coefficients dans le faisceau $\mathcal{T}_X$ de champs de vecteurs holomorphes de la normalisation non singulière X de $\overline{X}$ (Théorème 4.1).

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Introduction

An irreducible hypersurface $\overline{X}$ in the complex projective 4-space $\mathbb{P}^4(\mathbb{C})$ is called an *algebraic threefold with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbb{C}^4$ at every point of $\overline{X}$:

$$(0.1)\qquad \begin{cases} \text{(i)} & w=0 & \text{(simple point)}\\ \text{(ii)} & zw=0 & \text{(ordinary double point)}\\ \text{(iii)} & yzw=0 & \text{(ordinary triple point)}\\ \text{(iv)} & xyzw=0 & \text{(ordinary quadruple point)}\\ \text{(v)} & xy^2-z^2=0 & \text{(cuspidal point)}\\ \text{(vi)} & w(xy^2-z^2)=0 & \text{(stationary point)} \end{cases}$$

where (x,y,z,w) is the coordinate on $\mathbb{C}^4$. These singularities arise if we project a non-singular threefold embedded in a sufficiently higher dimensional complex projective space to its four dimensional linear subspace by a *generic* linear projection ([**R**]), though the singularities (iv) and (vi) above do not occur in the surface case. This fact can also be proved by use of the classification theory of multi-germs of *locally stable* holomorphic maps ([**M-3**], [**T-1**]). Indeed, in the threefold case, the pair of dimensions of the source and target manifolds belongs to the so-called *nice range*([**M-2**]). Hence the multi-germ of a *generic* linear projection at the inverse image of any point of $\overline{X}$ is *stable*, *i.e.*, stable under small deformations ([**M-4**]).

In [**T-2**] we have proved, for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points, a formula expressing the Euler number $\chi(X)$ of the non-singular normalization X of $\overline{X}$ in terms of numerical characteristics of $\overline{X}$ and its singular loci. Note that, by the Gauss-Bonnet formula, the Euler number $\chi(X)$ is equal to the Chern number $\int_X c_3$, where c_3 denotes the top Chern class of X. In § 1 we shall extend this formula to the general case where $\overline{X}$ admits quadruple points. In this general case, we need to blow up $\overline{X}$ twice. First, along the quadruple point locus, and secondly, along the triple point locus. It turns out that the existence of quadruple points adds only the term $4\#\Sigma\overline{q}$ to the formula, where $\#\Sigma\overline{q}$ denotes the cardinal number of the quadruple point locus $\Sigma\overline{q}$. Using Fulton-MacPherson's intersection theory, especially, the *excess intersection formula* ([**F**], Theorem 6.3, p. 102), the *blow-up formula* (ibid., Theorem 6.7, p. 116), the *double point formula* (ibid., Theorem 9.3, p. 166) and the *ramification formula* (ibid., Example 3.2.20, p. 62), we compute the *push-forwards* $f_*[D]^2$ and $f_*[D]^3$ for D the inverse image of the singular locus of $\overline{X}$ by the normalization map in order to know the *Segre classes* $s(\overline{J},\overline{X})_i$ $(0\leqslant i\leqslant 2)$ of the singular subscheme $\overline{J}$ defined by the Jacobian ideal of $\overline{X}$.

In § 2 we shall give a formula for the Chern number $\int_X c_1^3=-[K_X]^3$, where $[K_X]$ is the canonical class of X. The expressions for $f_*[D]^2$ and $f_*[D]^3$ obtained in § 1 enable us to compute it, because $[K_X]=f^*[\overline{X}+K_Y]-[D]$ by the *double point formula*, where

K_Y is the canonical divisor of $\mathbb{P}^4(\mathbb{C})$. In § 3 we shall give a formula for the Chern number $\int_X c_1c_2$. In fact, we shall calculate the Euler-Poincaré characteristic $\chi(X, K_X)$ with coefficient in the canonical line bundle of X, which is equal to $-(1/24)\int_X c_1c_2$ by the Riemann-Roch theorem. In § 4, as a by-product, we shall give a numerical formula for the Euler-Poincaré characteristic $\chi(X, \mathcal{T}_X)$ with coefficient in the sheaf $\mathcal{T}_X$ of holomorphic tangent vector fields on X.

Notation and Terminology

Throughout this article we fix the notation as follows:

$Y := \mathbb{P}^4(\mathbb{C})$: the complex projective 4-space,
$\overline{X}$: an algebraic threefold with ordinary singularities in Y,
$\overline{J}$: the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,
$\overline{D}$: the singular locus of $\overline{X}$,
$\overline{T}$: the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,
$\overline{C}$: the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points,
$\Sigma\overline{q}$: the quadruple point locus of $\overline{X}$,
$\Sigma\overline{s}$: the stationary point locus of $\overline{X}$,
$n_{\overline{X}} : X \to \overline{X}$: the normalization of $\overline{X}$,
$f : X \to Y$: the composite of the normalization map $n_{\overline{X}}$ and the inclusion $\bar{\iota} : \overline{X} \hookrightarrow Y$,
J: the scheme-theoretic inverse of $\overline{J}$ by f,
D, T, C and Σq: the inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{q}$ by f, respectively,
$\Sigma s = T \cap C$: the intersection of T and C.

We put

$$n := \deg \overline{X} \text{ (the degree of } \overline{X} \text{ in } \mathbb{P}^4(\mathbb{C})), \quad m := \deg \overline{D}, \quad t := \deg \overline{T}, \quad \gamma := \deg \overline{C}.$$

Note that $\overline{T}$ and $\overline{C}$ are non-singular curves, intersecting transversely at $\Sigma\overline{s}$, and that the normalization X of $\overline{X}$ is also non-singular. Calculating by use of local coordinates, we can easily see the following:

(i) J contains D, and the *residual scheme* to D in J is the reduced scheme C, *i.e.*, $\mathcal{I}_J = \mathcal{I}_D \otimes_{\mathcal{I}_X} \mathcal{I}_C$, where $\mathcal{I}_J$, $\mathcal{I}_D$, $\mathcal{I}_C$ are the ideal sheaves of J, D and C, respectively (*cf.* [**F**], Definition 9.2.1, p. 160);

(ii) D is a surface with ordinary singularities, whose singular locus is T,

(iii) D is the *double point* locus of the map $f : X \to Y$, *i.e.*, the closure of $\{q \in X \mid \# f^{-1}(f(q)) \geqslant 2\}$;

(iv) the map $f_{|D} : D \to \overline{D}$ is generically two to one, simply ramified at C;

(v) the map $f_{|T} : T \to \overline{T}$ is generically three to one, simply ramified at Σs.

Furthermore, we need the following diagram consisting of two fiber squares:

$$
\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "\tau_{T'}"'] & Y'' \arrow[d, "\sigma_{\overline{T}'}"] \\
X' \arrow[r, "f'"] \arrow[d, "\tau_{\Sigma q}"'] & Y' \arrow[d, "\sigma_{\Sigma\overline{q}}"] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}
\tag{0.2}
$$

which is defined as follows:

$\sigma_{\Sigma\overline{q}} : Y' \to Y$: the blowing-up of Y along the quadruple point locus $\Sigma\overline{q}$ of $\overline{X}$,

$\overline{X}'$: the proper inverse image of $\overline{X}$ by $\sigma_{\Sigma\overline{q}}$,

$X' := X \times_{\overline{X}} \overline{X}'$: the fiber product of X and $\overline{X}'$ over $\overline{X}$,

$n_{\overline{X}'} : X' \to \overline{X}'$: the projection to the second factor of $X \times_{\overline{X}} \overline{X}'$, which is nothing but the normalization of $\overline{X}'$,

$f' : X' \to Y'$: the composite of the normalization map $n_{\overline{X}'}$ and the inclusion $\overline{\iota}' : \overline{X}' \hookrightarrow Y'$,

Σq: the inverse image of the quadruple point locus $\Sigma\overline{q}$ of $\overline{X}$ by f,

$\tau_{\Sigma q} : X' \to X$: the projection to the first factor of $X \times_{\overline{X}} \overline{X}'$, which is nothing but the blowing-up of X along Σq,

$\overline{D}'$, $\overline{T}'$, $\overline{C}'$ and $\Sigma\overline{s}'$: the proper inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{s}$ by $\sigma_{\Sigma\overline{q}}$, respectively.

D', T' and C': the proper inverse images of D, T and C by $\tau_{\Sigma q}$, which are equal to the inverse images of $\overline{D}'$, $\overline{T}'$ and $\overline{C}'$ by f', respectively,

$\Sigma s'$: the inverse image of Σs by $\tau_{\Sigma q}$, which is equal to $T' \cap C'$,

$\sigma_{\overline{T}'} : Y'' \to Y'$: the blowing-up of Y' along $\overline{T}'$,

$\overline{X}''$: the proper inverse image of $\overline{X}'$ by $\sigma_{\overline{T}'}$,

$X'' := X' \times_{\overline{X}'} \overline{X}''$: the fiber product of X' and $\overline{X}''$ over $\overline{X}'$,

$n_{\overline{X}''} : X'' \to \overline{X}''$: the projection to the second factor of $X' \times_{\overline{X}'} \overline{X}''$, which is nothing but the normalization of $\overline{X}''$

$f'' : X'' \to Y''$: the composite of the normalization map $n_{\overline{X}''}$ and the inclusion $\iota'' : \overline{X}'' \hookrightarrow Y''$,

$\tau_{T'} : X'' \to X'$: the projection to the first factor of $X' \times_{\overline{X}'} \overline{X}''$, which is nothing but the blowing-up of X' along T',

$\overline{D}''$, $\overline{T}''$, $\overline{C}''$ and $\Sigma\overline{s}''$: the proper inverse images of $\overline{D}'$, $\overline{T}'$, $\overline{C}'$ and $\Sigma\overline{s}'$ by $\sigma_{\overline{T}}$, respectively,

D'', T'' and C'': the proper inverse images of D', T' and C' by $\tau_{T'}$, which are equal to the inverse images of $\overline{D}''$, $\overline{T}''$ and $\overline{C}''$ by f'', respectively,

$\Sigma s''$: the inverse image of $\Sigma s'$ by $\tau_{T'}$, which is equal to $T'' \cap C''$.

We also use the following notation throughout this article:

$[\alpha]$: the rational equivalence class of an algebraic cycle α,

$\alpha\cdot\beta$: the intersection class of two algebraic cycle classes $[\alpha]$ and $[\beta]$.

Finally, we give the definitions of *regular embeddings* and *local complete intersection morphisms* of schemes.

Definition 0.1. — We say a closed embedding $\iota : X \to Y$ of schemes is a *regular embedding of codimension d* if every point in X has an affine neighborhood U in Y, such that if A is the coordinate ring of U, I the ideal of A defining X, then I is generated by a regular sequence of length d.

If this is the case, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of X in Y, is a locally free sheaf of rank d. The *normal bundle* to X in Y, denoted by N_XY, is the vector bundle on X whose sheaf of sections is dual to $\mathcal{I}/\mathcal{I}^2$. Note that the normal bundle N_XY is canonically isomorphic to the normal cone C_XY for a (closed) regular embedding $\iota : X \to Y$ since the canonical map from $\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)$ to $S^{\cdot} := \Sigma_{k=0}^{\infty}\mathcal{I}^k/\mathcal{I}^{k+1}$ is an isomorphism (*cf.* [**F**], Appendix B, B.7).

Definition 0.2. — A morphism $f : X \to Y$ is called a *local complete intersection morphism of codimension d* if f factors into a (closed) regular embedding $\iota : X \to P$ of some constant codimension e, followed by a smooth morphism $p : P \to Y$ of constant relative dimension $d+e$.

1. The computation of $\int_X c_3$

In [**T-2**] we have proved, for an algebraic threefold $\overline{X}$ with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points, a formula expressing the the Euler number $\chi(X)$ of the non-singular normalization X of $\overline{X}$ in terms of numerical characteristics of $\overline{X}$ and its singular loci. We recall its proof briefly. We have first proved the following:

***Theorem 1.1* ([T-2], Theorem 2.1)**. — *We have a linear pencil $\overline{\mathcal{L}} := \bigcup_{\lambda\in\mathbb{P}^1}\overline{X_\lambda}$ on $\overline{X}$, consisting of hyperplane sections $\overline{X_\lambda}$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$, whose pull-back $\mathcal{L} := \bigcup_{\lambda\in\mathbb{P}^1}X_\lambda$ to X by the normalization map $f : X \to \overline{X}$ has the following properties: There exists a finite set $\{\lambda_1,\dots,\lambda_c\}$ of points of $\mathbb{P}^1$ such that*

(i) *X_λ is non-singular for λ with $\lambda\neq\lambda_i$ $(1\leqslant i\leqslant c)$, and*

(ii) *X_{λ_i} is a surface with only one isolated ordinary double point which is contained in $X\smallsetminus f^{-1}(\overline{C_\infty})$ for any i with $1\leqslant i\leqslant c$,*

where c is the class *of $\overline{X}$, i.e., the degree of the top polar class $[M_3]$ of $\overline{X}$ in $\mathbb{P}^4(\mathbb{C})$ (cf. [**P**]), and $\overline{C_\infty}$ the base point locus of the linear pencil $\overline{\mathcal{L}}$, which is an irreducible curve with m $(=\deg\overline{D})$ ordinary double points in $\mathbb{P}^2(\mathbb{C})$ whose degree is equal to n $(=\deg\overline{X})$.*