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ON SEMI-STABLE, SINGULAR CUBIC SURFACES

by

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Abstract. — This paper deals with semi-stable and stable singular cubic surfaces from the point of view of the geometric invariant theory. We are interested in properties of the subsets $i\mathcal{A}_1 j\mathcal{A}_2$ corresponding to all semi-stable, singular cubic surfaces with exactly i singular points of type A_1 and j singular points of type A_2. We consider semi-stable cubic surfaces as "csurfaces" of 6-point schemes in almost general position with some conditions of configurations. This is a generalization of the blowing-up of $\mathbb{P}^2$ at 6 points in general position. From relevant configurations of 6-point schemes, we can determine number of star points, the configuration of singular points, of lines and tritangent planes with multiplicities on semi-stable, singular cubic surfaces.

***Résumé* (Sur les surfaces cubiques semi-stables).** — Cet article concerne les surfaces cubiques semi-stables et stables du point de vue de la théorie géométrique des invariants. Nous nous sommes intéressé aux propriétés des sous-ensembles $i\mathcal{A}_1 j\mathcal{A}_2$ correspondant à toutes les surfaces cubiques singulières semi-stables avec exactement i points singuliers de type A_1 et j points singuliers de type A_2. Nous considérons les surfaces cubiques semi-stables comme « c-surfaces » d'ensembles de 6 points en position presque générale avec certaines conditions de configurations. Ceci est une généralisation de l'éclatement de $\mathbb{P}^2$ en 6 points en position générale. À partir de configurations adaptées d'ensembles de 6 points, nous pouvons déterminer le nombre de points « étoile », la configuration des points singuliers, des droites et des plans « tritangents » avec multiplicités sur les surfaces singulières cubiques semi-stables.

1. Introduction

Consider $\mathbb{P}^{19}$ as a parametrizing space of cubic surfaces in $\mathbb{P}^3_k$, where k is an algebraically closed field with characteristic 0. We have the action of PGL(4) on $\mathbb{P}^{19}$. The locus $\Delta \subset \mathbb{P}^{19}$ of singular cubic surfaces is a closed subset of codimension 1. Some

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classifications of singular cubic surfaces can be found in [**4**] or [**5**]. We are interested in singular cubic surfaces which correspond to semi-stable and stable points under the action of PGL(4) on $\mathbb{P}^{19}$ in the sense of the geometric invariant theory. One reason we are interested in these kinds of singularities is that the quotient space of semi-stable points over PGL(4) exists and it is a compactification of the moduli space of non-singular cubic surfaces.

It is well-known that the blowing-up of $\mathbb{P}^2$ at 6 points in general position is isomorphic to a non-singular cubic surface. Conversely, any non-singular cubic surface can be obtained in that way. A question arises naturally: is there a similar correspondence between a semi-stable, singular cubic surface and a 6-point scheme in some relevant configuration of its points? Showing such a correspondence is one of main goals of this paper. Namely, let X be a semi-stable cubic surface. Then there exists a 6*-point scheme* $\mathcal{P}$ such that the linear system $\mathcal{L}_{\mathcal{P}}$ of cubic forms in four variables through $\mathcal{P}$ has dimension 4; furthermore, for any basis of $\mathcal{L}_{\mathcal{P}}$, the closure of the image of the rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$ defined by the basis is a surface which is isomorphic to X. In this case, we have a morphism $Y \longrightarrow X$, where Y is the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$. In general, this is a blowing-down and not an isomorphism. A close study of such 6-point schemes enables us to determine the number of lines, the number of singularities of X and their configuration as well.

This also gives a way to compute the multiplicity of lines and tritangent planes on semi-stable, singular cubic surfaces. This investigation shows a clear picture on the configuration of lines and tritangent planes of semi-stable, singular cubic surfaces. Moreover, we will give definitions of *star point* and *proper star point* which are generalizations of the concept of Eckardt point on non-singular cubic surfaces. We will determine the number of (proper) star points on a general one of any class of semi-stable cubic surfaces and study some properties.

2. Stable and semi-stable, singular cubic surfaces

We denote by $i\mathcal{A}_1 j\mathcal{A}_2$ the subset of $\mathbb{P}^{19}$ corresponding to irreducible cubic surfaces with exactly i singular points of type A_1 and j singular points of type A_2. We refer to [**1**] and [**2**] or to [**4**] for general definitions of types of singularities. We will see later that these subsets correspond to all semi-stable, singular cubic surfaces with respect to the action of PGL(4) on $\mathbb{P}^{19}$.

Remark 2.1

(i) In the case of cubic surfaces, the singularities of types A_1 and A_2 are characterized as follows. A point P on a cubic surface with only isolated singularities is called a *singular point of type* A_1 (respectively A_2) if the tangent cone at P is an irreducible quadric surface (respectively if the tangent cone at P consists of two distinct planes whose intersection line does not lie on the surface).

(ii) We have $2i+3j \leqslant 9$, $i \leqslant 4$ and $(i,j) \neq (3,1)$, see [**4**], p. 255 or [**11**], pp. 49-50. We use $j\mathcal{A}_2$ and $i\mathcal{A}_1$ instead of $0\mathcal{A}_1 j\mathcal{A}_2$ and $i\mathcal{A}_1 0\mathcal{A}_2$, respectively.

(iii) By the definition, a semi-stable, singular cubic surface can be given by a polynomial in the following form:

$$F = x_3 f_2(x_0,x_1,x_2) + f_3(x_0,x_1,x_2),$$

where f_i for $i=1,2$ is a homogeneous polynomial of degree i. Then the type of singularity of the surface is characterized by $\operatorname{rank}(f_2)$ and the configuration of points in $V_{\mathbb{P}^2}(f_2,f_3)$.

Some interesting properties of subsets $i\mathcal{A}_1 jA_2$ are shown in the following.

Proposition 2.2. — *The subsets $i\mathcal{A}_1 jA_2$ are irreducible of codimension $i+2j$ in $\mathbb{P}^{19}$ and have a relation as shown in the Figure 1, where $A \longrightarrow B$ means that $\overline{A} \subset \overline{B}$ and subsets are in the same column iff they have the same codimension.*

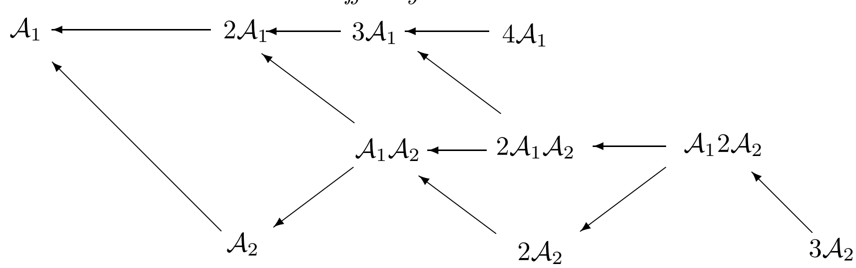

FIGURE 1

Proof. — This follows from [**3**], Prop. 2.1. and [**3**], Fig. 1, p. 435. ☐

Proposition 2.3. — *On the action of* $\mathrm{PGL}(4)$ *on $\mathbb{P}^{19}$, we have:*

(i) *The subset of stable points consists of points in $\mathbb{P}^{19}-\Delta$ and those of types $i\mathcal{A}_1$ for $1 \leqslant i \leqslant 4$.*

(ii) *The subset of semi-stable points consists of points in $\mathbb{P}^{19}-\Delta$ and all those of types $i\mathcal{A}_1 jA_2$.*

Proof. — This result was mentioned, for instance, in [**10**], p. 80 or [**9**], p. 51. A detailed proof could be found in [**11**], 3.2.14. ☐

3. Semi-stable as csurfaces of 6-point schemes in almost general position

As in the case of non-singular cubic surfaces, we show that each semi-stable, singular cubic surface corresponds to a relevant 6-point scheme in almost general position. Moreover we prove that the corresponding semi-stable cubic surfaces are isomorphic if their 6-points schemes are different by quadratic transformations.

Definition. — A 6-*point scheme* is a closed subscheme in $\mathbb{P}^2$ of dimension zero and of length 6. Any 6-point scheme $\mathcal{P}$ defines a formal cycle $c(\mathcal{P}) = \sum n_i P_i$ for $\sum n_i = 6$; the set of the points P_i is called *the support* of $\mathcal{P}$ and denoted by $\mathrm{Supp}(\mathcal{P})$. If the linear system of all cubic forms passing through a 6-point scheme $\mathcal{P}$ has (linear) dimension 4, then $\mathcal{P}$ is called a 6-point scheme *in almost general position.*

Let Hilb_n denote the Hilbert scheme of zero-dimensional closed subschemes of length n in $\mathbb{P}^2$. We denote by $\mathcal{H}^a$ the subscheme of Hilb_6 consisting of all 6-point schemes in almost general position.

Let $\mathcal{P} \in \mathcal{H}^a$ and let l be any line in $\mathbb{P}^2$ such that $l \cap \mathcal{P} \neq \varnothing$. Then the length of $l \cap \mathcal{P}$ is not greater than 4.

Definition. — Let $\mathcal{P} \in \mathcal{H}^a$. We say that $\mathcal{P}$ is a 6-point scheme with *no 4 points on a line* if there does not exist any line l in $\mathbb{P}^2$ such that the length of $l \cap \mathcal{P}$ is equal to 4. Denote by $\mathcal{H}^o$ the subset of 6-point schemes with no 4 points on a line.

Lemma 3.1. — *Let $\mathcal{P} \in \mathcal{H}^o$. Let $\mathcal{L}_\mathcal{P}$ be the linear system of cubic forms passing through $\mathcal{P}$.*

(i) *The base locus of $\mathcal{L}_\mathcal{P}$ is the support of $\mathcal{P}$.*

(ii) *Let $\{f_1, \ldots, f_4\}$ be a basis of $\mathcal{L}_\mathcal{P}$. Consider the morphism*

$$\begin{aligned} \psi : \mathbb{P}^2 - \mathrm{Supp}(\mathcal{P}) &\longrightarrow \mathbb{P}^3 \\ P \qquad &\longmapsto (f_1(P) : f_2(P) : f_3(P) : f_4(P)). \end{aligned}$$

Let X be the closure of the image of ψ. Then X is a cubic surface.

(iii) *If $\{g_1, \ldots, g_4\}$ is another basis of $\mathcal{L}_\mathcal{P}$ and X' is the cubic surface obtained as in (ii), then X and X' are isomorphic.*

Proof

(i) Let $P \in \mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$. Since $\mathcal{P}$ does not have 4 points on a line, there exists a cubic form in $\mathcal{L}_\mathcal{P}$ which does not contain P. This implies that the base locus of $\mathcal{L}_\mathcal{P}$ is the support of $\mathcal{P}$.

(ii) Let Q_1, Q_2 be two general points in $\mathbb{P}^2 - \mathrm{Supp}(\mathcal{P})$. The linear subspaces consisting of cubic forms through $\mathcal{P} \cup \{Q_1\}$ and $\mathcal{P} \cup \{Q_1, Q_2\}$ respectively have dimension 3 and 2. This implies that there exists a cubic form in $\mathcal{L}_\mathcal{P}$ which contains Q_1 but does not contain Q_2 and conversely. This means that ψ is injective over an open subset of $\mathbb{P}^2$. Moreover, any two general cubic forms in $\mathcal{L}_\mathcal{P}$ have 3 other points in common which do not belong to $\mathcal{P}$. This implies that X is a cubic surface.

(iii) Let $A = (a_{ij})_{4\times 4}$ be the base change matrix from $\{f_1, \ldots, f_4\}$ to $\{g_1, \ldots, g_4\}$. Then A defines a projective transformation which transforms X to X'. □

Definition. — A *csurface* is an algebraic variety Y such that there exists a cubic surface $X \subset \mathbb{P}^3$ such that $X \cong Y$. From the lemma, we see that each $\mathcal{P} \in \mathcal{H}^o$ determines uniquely (up to isomorphisms) a csurface, which is called *the csurface* *of* $\mathcal{P}$. If $\mathcal{P}$ consists of 6 points in general position, then the csurface of $\mathcal{P}$ is the blowing-up of $\mathbb{P}^2$ at $\mathcal{P}$.

Definition. — Let $P_0 = (1:0:0)$, $P_1 = (0:1:0)$ and $P_2 = (0:0:1)$. Let $\varphi : \mathbb{P}^2 - - \to \mathbb{P}^2$ be the quadratic transformation with respect to P_0, P_1 and P_2 (see [**8**], V.4.2.3). Let C be the cubic curve given by

$$(1) \qquad F = \sum a_{ijk}\, x_0^i x_1^j x_2^k \text{ for } i+j+k = 3 \text{ and } 0 \leqslant i,j,k \leqslant 2.$$

The cubic curve defined by $F_\varphi := \sum a_{ijk}\, y_0^{2-i} y_1^{2-j} y_2^{2-k}$ in $\mathbb{P}^2$ is called *the image of* C *by* φ and is denoted by C_φ.

Lemma 3.2. — *Let* $\mathcal{P} \in \mathcal{H}^o$. *Suppose that* $\mathrm{Supp}(\mathcal{P})$ *contains 3 distinct points* P_1, P_2 *and* P_3. *Suppose further that there exists a cubic form in* $\mathcal{L}_\mathcal{P}$ *which is non-singular at any* P_i *for* $i = 1,2,3$. *Let* φ *be the quadratic transformation with respect to* P_1, P_2 *and* P_3. *Then the set* $\varphi(\mathcal{L}_\mathcal{P}) := \{F\varphi \mid F \in \mathcal{L}_\mathcal{P}\}$ *is a 4-dimensional linear space whose base locus is of dimension* 0.

Proof. — Choose coordinates such that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$ and $P_3 = (0:0:1)$. Suppose that the base locus of $\varphi(\mathcal{L}_\mathcal{P})$ contains an irreducible component Y of positive dimension. Since φ is one-to-one in $\mathbb{P}^2 - V(x_0x_1x_2)$, the variety Y is contained in $V(y_0y_1y_2)$. Assume that Y contains the line $d_{12} = V(y_0)$. This means that for any $F \in \mathcal{L}_\mathcal{P}$, we have $F_\varphi = y_0 g_2(y_0, y_1, y_2)$ where g_2 is a homogeneous polynomial of degree 2 and vanishes at $Q_3 = (0:0:1)$. Then $F = (F_\varphi)_{\varphi^{-1}}$ is singular at $P_1 = (1:0:0)$. A contradiction! □

Definition. — Let $\mathcal{P} \in \mathcal{H}^o$ satisfy the conditions as in the previous lemma. Let I be the ideal generated by all cubic forms in $\varphi(\mathcal{L}_\mathcal{P})$. The scheme defined by this ideal is called *the image of* $\mathcal{P}$ and denoted by $\varphi(\mathcal{P})$.

Proposition 3.3. — *Every semi-stable cubic surface is isomorphic to the csurface of some* 6*-point scheme in almost general position with no 4 points on a line.*

Proof. — Let X be a semi-stable cubic surface. If X is a non-singular cubic surface then it is isomorphic to the blowing-up of a 6-point scheme in general position. We consider the case that X is singular.

Suppose that X does not have any A_2 singularity. By choosing coordinates, we may assume X to be defined by

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i for $i = 2, 3$ is a homogeneous polynomial of degree i and f_2 is irreducible. The scheme $\mathcal{P} = V_{\mathbb{P}^2}(f_2, f_3)$ defines an element in $\mathcal{H}^o$. The 6-point scheme $\mathcal{P}$ is contained in an irreducible conic curve defined by f_2 and the cycle $c(\mathcal{P})$ corresponds to a partition $(2^{i-1}1^k)$ of 6. Let $\mathcal{L}_\mathcal{P}$ be the linear space of cubic forms passing through $\mathcal{P}$. Since $\mathcal{P}$ does not contain any triple point, we see that the cubic forms