Séminaires & Congrès
10, 2005, p. 443–460

ISOLATED CRITICAL POINTS AND ADIABATIC LIMITS OF CHERN FORMS

by

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Dedicated to Professor Tatsuo Suwa on his 60th birthday

Abstract. — In this note, we compute the adiabatic limit of Chern forms for holomorphic fibrations over complex curves. We assume that the projection of the fibration has only isolated critical points.

***Résumé* (Points critiques isolés et limites adiabatiques des formes de Chern).** — Dans cet article, nous calculons la limite adiabatique des formes de Chern pour les fibrations holomorphes sur des coubes complexes. Nous supposons que le projection de la fibration n'a que des points critiques isolés.

1. Introduction

Let X be a complex manifold of dimension $n+1$ and S a Riemann surface. Let $f : X \to S$ be a proper surjective holomorphic map. The critical locus of f is the analytic subset of X defined by

$$\Sigma_f = \{p \in X\,;\, df_p = 0\}.$$

In this note, we always assume that Σ_f is discrete.

Let g^{TX} be a Hermitian metric on the holomorphic tangent bundle TX. Let g^{TS} be a Hermitian metric on TS. Define the family of Hermitian metrics on TX by

$$g_\varepsilon^{TX} = g^{TX} + \frac{1}{\varepsilon^2} f^* g^{TS} \qquad (\varepsilon > 0).$$

Let $\nabla^{TX,g_\varepsilon^{TX}}$ be the holomorphic Hermitian connection of (TX, g_ε^{TX}), whose curvature form is denoted by $R^{TX,g_\varepsilon^{TX}}$. Then $R^{TX,g_\varepsilon^{TX}}$ is a $(1,1)$-form on X with values in $\mathrm{End}(TX)$. Let $c_i(TX, g_\varepsilon^{TX})$ be the i-th Chern form of (TX, g_ε^{TX}).

2000 *Mathematics Subject Classification.* — 58K05, 58K20, 57R20, 57R70, 58A25.
Key words and phrases. — Isolated critical point, adiabatic limit, Chern form, Milnor number.

Research partially supported by the Grants-in-Aid for Scientific Research (B)(2): 14740035, JSPS.

Let $P(c) = P(c_1, \ldots, c_{n+1}) \in \mathbf{C}[c_1, \ldots, c_{n+1}]$ be a polynomial in the variables $c_1, \ldots, c_{n+1}$. The purpose of this note is to study the family of differential forms $P(TX, g_\varepsilon^{TX}) := P(c(TX, g_\varepsilon^{TX}))$ as $\varepsilon \to 0$, called the *adiabatic limit*, under certain assumptions on the metrics g^{TX}, g^{TS} (see Assumption 2.1).

The study of this problem was initiated by Bismut and Bost in [**3**, Sect. 6 (a)]; they treated the case where $\dim X = 2$, the map f has only non-degenerate critical points, and $P(c)$ is the Todd polynomial. They applied their formula for the adiabatic limit to compute the holonomy of the determinant line bundles on S ([**3**, Sect. 6 (b), (c)]). Then Bismut treated in [**2**, Sect. 1 (e)] the case where $\dim X$ is arbitrary, the critical locus of the map f is locally defined by the equation $f(z_0, z_1, z') = z_0 z_1$, and $P(c)$ is arbitrary; he used his result to study the boundary behavior of Quillen metrics.

The goal of this note is to establish the convergence of the adiabatic limit $\lim_{\varepsilon\to 0} P(TX, g_\varepsilon^{TX})$ in the sense of currents on X and to compute the explicit formula for it. In particular, we extend [**3**, Sect. 6 (a)] to the case where f has only isolated critical points. Our result (Theorem 2.2) is compatible with [**15**].

2. Statement of the Result

Let $f : X \to S$ be a proper surjective holomorphic map between complex manifolds. Throughout this note, we assume the following:

(i) The critical locus Σ_f is a discrete subset of X.
(ii) $\dim X = n + 1$ and $\dim S = 1$.

Let g^{TX} and g^{TS} be Hermitian metrics on TX and TS, respectively. We define the family of Hermitian metrics $\{g_\varepsilon^{TX}\}_{\varepsilon>0}$ by

$$g_\varepsilon^{TX} := g^{TX} + \varepsilon^{-2} f^* g^{TS}.$$

The unit disc $\{s \in \mathbf{C};\, |s| < 1\}$ and the unit punctured disc $\{s \in \mathbf{C};\, 0 < |s| < 1\}$ are denoted by Δ and $\Delta^* = \Delta \setminus \{0\}$, respectively.

2.1. Assumptions on metrics.

— Let $\Gamma_f \subset X \times S$ be the graph of f:

$$\Gamma_f = \{(x, t) \in X \times S;\, f(x) = t\}.$$

Let $\mathrm{pr}_1 : \Gamma_f \to X$ and $\mathrm{pr}_2 : \Gamma_f \to S$ be the natural projections. Let $(U_p, (z_0, \ldots, z_n))$ be a coordinate neighborhood of $p \in \Sigma_f$ in X centered at p. Let $(D_{f(p)}, t)$ be a coordinate neighborhood of $f(p)$ in S centered at $f(p)$. Assume that

(i) $U_p \cap U_q = \varnothing$ for $p, q \in \Sigma_f$ with $p \neq q$;
(ii) $(U_p, p) \cong (\Delta^{n+1}, 0)$;
(iii) $(f(U_p), f(p)) \subset (D_{f(p)}, 0)$.

Then $\Gamma_f|_{U_p}$ is a submanifold of $U_p \times D_{f(p)}$. Let $\iota : \Gamma_f|_{U_p} \hookrightarrow U_p \times D_{f(p)}$ be the inclusion. We have the commutative diagram:

$$\begin{array}{ccc} (\Gamma_f|_{U_p},\, (p, f(p))) & \xrightarrow{\ \iota\ } & (U_p \times D_{f(p)},\, (0,0)) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ (U_p, p) & \xrightarrow{\quad f \quad} & (D_{f(p)}, 0). \end{array}$$

Assumption 2.1. — Let $\delta \geqslant 0$ be a constant. Assume that the Hermitian metrics g^{TX} and g^{TS} are expressed as follows on each U_p $(p \in \Sigma_f)$:

$$\mathrm{pr}_1^* g^{TX}|_{(\Gamma_f|_{U_p})} = \left\{ \sum_i dz_i \otimes d\overline{z}_i + \delta \cdot dt \otimes d\overline{t} \right\}\Bigg|_{(\Gamma_f|_{U_p})}, \tag{1}$$

$$g^{TS}|_{D_{f(p)}} = dt \otimes d\overline{t}. \tag{2}$$

We are mainly interested in the case $\delta = 0$ because $g^{TX}|_{U_p}$ is the restriction of the Euclidean metric on $\mathbf{C}^{n+1}$ in this case.

2.2. Chern forms. — Let $M_{n+1}(\mathbf{C})$ be the set of all complex $(n+1) \times (n+1)$ matrices. For $A \in M_{n+1}(\mathbf{C})$, set $c(A) = \det(I_{n+1} + A) = 1 + c_1(A) + \cdots + c_{n+1}(A)$, where $c_i(A)$ is homogeneous of degree i. For a polynomial $P(c) = P(c_1, \ldots, c_{n+1}) \in \mathbf{C}[c_1, \ldots, c_{n+1}]$, set $P(A) = P(c_1(A), \ldots, c_{n+1}(A))$.

Denote by $A_X^{p,q}$ (resp. A_X^r) the vector space of smooth (p,q)-forms (resp. r-forms) on X. For a complex vector bundle F on X, the set of smooth (p,q)-forms on X with values in F is denoted by $A_X^{p,q}(F)$. For $\Phi \in A_X^*$, Φ^{top} denotes the bidegree $(\dim X, \dim X)$-part of Φ. Hence $\Phi^{\mathrm{top}} \in A_X^{n+1,n+1}$.

Let (E, h^E) be a holomorphic Hermitian vector bundle on X. Let ∇^{E,h^E} be the holomorphic Hermitian connection. Namely, the $(0,1)$-part of ∇^{E,h^E} is given by the $\overline{\partial}$-operator and ∇^{E,h^E} is compatible with the metric h^E (*cf.* [**10**, Chap. 1, Sect. 4]). Let $R^{E,h^E} = (\nabla^{E,h^E})^2 \in A_X^{1,1}(\mathrm{End}(E))$ be the curvature form of ∇^{E,h^E}. Set

$$c(E, h^E) = \sum_{i=0}^{\mathrm{rank}(E)} c_i(E, h^E) := c\left(\frac{i}{2\pi} R^{E,h^E}\right) \in \bigoplus_{p \geqslant 0} A_X^{p,p}.$$

2.3. The convergence of adiabatic limits. — Let

$$Tf := \ker\{f_* : TX|_{X \smallsetminus \Sigma_f} \longrightarrow f^*TS\}$$

be the relative holomorphic tangent bundle of the map $f : X \to S$. Then Tf is a holomorphic subbundle of $TX|_{X \smallsetminus \Sigma_f}$.

Let $g^{Tf} = g^{TX}|_{Tf} = (g_\varepsilon^{TX})|_{Tf}$ be the Hermitian metric on Tf induced from g_ε^{TX}. Then g^{Tf} is independent of $\varepsilon > 0$. Let $R^{Tf,g^{Tf}}$ be the curvature of (Tf, g^{Tf}). The i-th Chern form $c_i(Tf, g^{Tf})$ lies in $A_{X \smallsetminus \Sigma_f}^{i,i}$ for $i = 1, \ldots, n$.

For $p \in \Sigma_f$, let $\mu(f,p) \in \mathbf{N}$ be the *Milnor number* of f at p, *i.e.*,

$$\mu(f,p) := \dim_{\mathbf{C}} \mathbf{C}\{z_0,\dots,z_n\} \Big/ \left(\frac{\partial f}{\partial z_0}(z),\dots,\frac{\partial f}{\partial z_n}(z)\right),$$

where $(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}) \subset \mathbf{C}\{z_0,\dots,z_n\}$ is the ideal generated by the germs $\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}$.

The *Dirac δ-current* supported at $p \in \Sigma_f$ is the $(n+1,n+1)$-current δ_p on X defined by

$$\int_X \varphi\, \delta_p := \varphi(p), \qquad \forall\, \varphi \in C_0^\infty(X).$$

For a formal power series of one variable $\varphi(t) \in \mathbf{C}[[t]]$, let $\varphi(t)|_{t^m}$ be the coefficient of the term t^m in $\varphi(t)$, *i.e.*, $\varphi(t)|_{t^m} = \frac{1}{m!}\left(\frac{d}{dt}\right)^m|_{t=0}\,\varphi(t)$.

Main Theorem 2.2. — *With the same notation as above, assume that Σ_f is a discrete subset of X and that the metrics g^{TX}, g^{TS} verify Assumption* 2.1. *Then the following hold:*

(1) *The differential form $P(Tf \oplus f^*TS,\, g^{Tf} \oplus f^*g^{TS})^{\mathrm{top}} \in A^{n+1,n+1}_{X\smallsetminus\Sigma_f}$ extends trivially to a smooth $(n+1,n+1)$-form on X.*

(2) *The adiabatic limit $\lim_{\varepsilon\to 0} P(TX, g^{TX}_\varepsilon)^{\mathrm{top}}$ converges to a $(n+1,n+1)$-current on X. Moreover, the following identity holds:*

$$\lim_{\varepsilon\to 0} P(TX, g^{TX}_\varepsilon)^{\mathrm{top}} = P(Tf \oplus f^*TS,\, g^{Tf} \oplus f^*g^{TS})^{\mathrm{top}} + P(-t,\dots,(-t)^{n+1})|_{t^{n+1}} \cdot \sum_{p\in\Sigma_f} \mu(f,p)\,\delta_p, \tag{2.1}$$

In particular, the following equation of currents on U_p holds:

$$\lim_{\varepsilon\to 0} P(TX, g^{TX}_\varepsilon)^{\mathrm{top}}|_{U_p} = P(-t,\dots,(-t)^{n+1})|_{t^{n+1}} \cdot \mu(f,p)\,\delta_p. \tag{2.2}$$

Corollary 2.3 **([8], [4, Example 14.1.5], [7, Chap. VI, 3], [9, Cor. 2.4])**

Let X be a compact complex manifold of dimension $n+1$ and S a compact Riemann surface. Let $f : X \to S$ be a proper surjective holomorphic map with general fiber F. Let $\chi_{\mathrm{EP}}(X)$, $\chi_{\mathrm{EP}}(F)$, $\chi_{\mathrm{EP}}(S)$ be the topological Euler-Poincaré numbers of X, F, S, respectively. If Σ_f is a finite set, then the following identity holds:

$$\chi_{\mathrm{EP}}(X) = \chi_{\mathrm{EP}}(F)\chi_{\mathrm{EP}}(S) + (-1)^{n+1} \sum_{p\in\Sigma_f} \mu(f,p).$$

Proof of Corollary 2.3. — Consider the polynomial $P(A) = c_{n+1}(A) = \det(A)$. Then the corresponding genus is the Euler characteristic. Since

$$c_{n+1}(Tf \oplus f^*TS, g^{Tf} \oplus f^*g^{TS}) = c_n(Tf, g^{Tf}) \wedge f^*c_1(TS, g^{TS}) \in A_X^{n+1,n+1}$$

by Theorem 2.2 (1), the result follows from (2.1) and the projection formula:

$$\int_X c_{n+1}(Tf \oplus f^*TS, g^{Tf} \oplus f^*g^{TS}) = \int_F c_n(Tf, g^{Tf})|_F \int_S c_1(TS, g^{TS}) = \chi_{\mathrm{EP}}(F)\chi_{\mathrm{EP}}(S).$$

□

Example 2.4. — Let A be an Abelian variety of dimension g and E an elliptic curve. Let $X \subset A \times E$ be a smooth hypersurface such that the restriction of the projection $\mathrm{pr}_2|_X : X \to E$ has only isolated critical points. Set $f = \mathrm{pr}_2|_X$.

Let g^{TA} and g^{TE} be the *flat* Kähler metrics on TA and TE, respectively. For $\varepsilon > 0$, set

$$g^{TX}_{\varepsilon} = g^{TA} \oplus \left(1 + \frac{1}{\varepsilon^2}\right) g^{TE}\Big|_X .$$

Then, for all $x \in X$, there is a neighborhood U_x in $A \times E$ such that the metrics $g^{TX} := g^{TX}_{\infty}$ and g^{TE} verify Assumption 2.1 on U_x. The first term of the R.H.S. of (2.1) vanishes identically on X by Propositions 4.1 and 4.2 below. Hence it follows from (2.1) that

$$\lim_{\varepsilon \to 0} P(TX, g^{TX}_{\varepsilon})^{\mathrm{top}} = P(-t, \ldots, (-t)^g)|_{t^g} \cdot \sum_{p \in \Sigma_f} \mu(f,p)\, \delta_p. \tag{2.3}$$

In particular, the support of the adiabatic limit $\lim_{\varepsilon \to 0} P(TX, g^{TX}_{\varepsilon})^{\mathrm{top}}$ concentrates on the critical locus Σ_f in this example.

Remark 2.5. — We can verify (2.3) as an identity of cohomology classes as follows. Let N be the normal bundle of X in $A \times E$. Then we have the exact sequence of holomorphic vector bundles on X:

$$0 \longrightarrow TX \longrightarrow T(A \times E)|_X = \mathbf{C}^{g+1} \longrightarrow N \longrightarrow 0,$$

from which we obtain $c(X) = c(N)^{-1} = (1 + c_1(N))^{-1}$. Hence $c_i(X) = (-c_1(N))^i$ for $i = 1, \ldots, g$ and

$$P(c(X)) = P(-t, \ldots, (-t)^g)|_{t^g} \cdot c_1(N)^g = (-1)^g P(-t, \ldots, (-t)^g)|_{t^g} \cdot c_g(X).$$

Since $\chi_{\mathrm{EP}}(E) = 0$, this yields that

$$\begin{aligned}
\int_X P(c(X)) &= (-1)^g P(-t, \ldots, (-t)^g)|_{t^g} \cdot \chi_{\mathrm{EP}}(X) \\
&= (-1)^g P(-t, \ldots, (-t)^g)|_{t^g} \cdot \Big\{ \chi_{\mathrm{EP}}(F)\chi_{\mathrm{EP}}(E) + (-1)^g \sum_{p \in \Sigma_f} \mu(f,p) \Big\} \\
&= P(-t, \ldots, (-t)^g)|_{t^g} \cdot \sum_{p \in \Sigma_f} \mu(f,p).
\end{aligned}$$

3. An analytic characterization of the Milnor number

Set $U := \Delta^{n+1}$. We denote by $z = (z_0, \ldots, z_n)$ the system of coordinates of U. Let $f : (U, 0) \to (\mathbf{C}, 0)$ be a holomorphic function on U such that

$$\Sigma_f = \{0\}.$$