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ON THE CONNECTION BETWEEN AFFINE AND PROJECTIVE FUNDAMENTAL GROUPS OF LINE ARRANGEMENTS AND CURVES

by

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Abstract. — In this note we prove a decomposition related to the affine fundamental group and the projective fundamental group of a line arrangement and a reducible curve with a line component. We give some applications to this result.

Résumé **(Sur le rapport entre les groupes fondamentaux d'arrangements affine et projectifs de droites et de courbes)**

Dans cet article, nous montrons une décomposition reliée au groupe fondamental affine et au groupe fondamental projectif d'un arrangement de droites et d'une courbe réductible avec une composante linéaire. Nous donnons quelques applications de ce résultat.

1. Introduction

The fundamental group of complements of plane curves is a very important topological invariant with many different applications. There are two such invariants: the *affine fundamental group* of a plane curve, which is the fundamental group of its affine complement, and its *projective fundamental group*, which is the fundamental group of its projective complement.

Oka [**7**] has proved the following interesting result, which sheds new light on the connection between these two fundamental groups:

***Theorem 1.1* (Oka).** — *Let C be a curve in $\mathbb{CP}^2$ and let L be a general line to C, i.e. L intersects C in only simple points. Then, we have a central extension:*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\mathbb{CP}^2 - (C \cup L)) \longrightarrow \pi_1(\mathbb{CP}^2 - C) \longrightarrow 1$$

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Since L is a general line to C, then:

$$\pi_1(\mathbb{CP}^2 - (C \cup L)) = \pi_1(\mathbb{C}^2 - C).$$

Hence, we get the following interesting connection between the two fundamental groups:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\mathbb{C}^2 - C) \longrightarrow \pi_1(\mathbb{CP}^2 - C) \longrightarrow 1$$

A natural question is:

Question 1.2. — *Under which conditions does this short exact sequence split? Notice that when it does, we have the following decomposition:*

$$\pi_1(\mathbb{C}^2 - C) \cong \pi_1(\mathbb{CP}^2 - C) \oplus \mathbb{Z}$$

A *real line arrangement* in $\mathbb{C}^2$ is a finite union of copies of $\mathbb{C}$ in $\mathbb{C}^2$, whose equations can be written by real coefficients. Some families of real line arrangements were already proved to satisfy this condition, see [**2**].

Here, we show that:

Theorem 1.3. — *If $\mathcal{L}$ is a real line arrangement, then such a decomposition holds:*

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}) \oplus \mathbb{Z}$$

Actually, one can see that the same argument holds for arbitrary line arrangements (see Theorem 2.3). Moreover, we give a different condition for this decomposition to hold: If C is a plane curve with a line component, *i.e.* $C = C' \cup L$ where L is a line, we have that $\pi_1(\mathbb{C}^2 - C) = \pi_1(\mathbb{CP}^2 - C) \oplus \mathbb{Z}$ too.

These results can be applied to the computation of the affine fundamental groups of line arrangements, since the projective fundamental group is an easier object to deal with than the affine fundamental group.

The paper is organized as follows. In Section 2 we prove Theorem 1.3. The other condition is discussed in Section 3, and in the last section we give some applications.

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. — Let $\mathcal{L}$ be a real line arrangement with n lines. Let L be an arbitrary line which intersects $\mathcal{L}$ transversally. By the following remark, one can reduce the proof to the case where L is a line which intersects transversally all the lines in $\mathcal{L}$ and all the intersection points of L with lines in $\mathcal{L}$ are to the left of all the intersection points of $\mathcal{L}$ (see Figure 1 for such an example, where the arrangement $\mathcal{L}$ consists of L_1, L_2, L_3 and L_4).

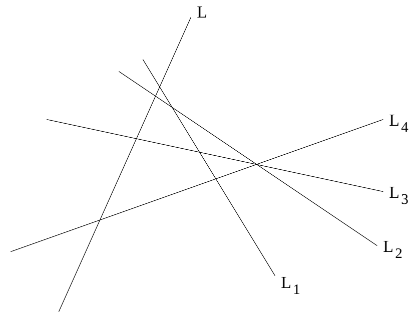

FIGURE 1. An example

Remark 2.1. — We have proved in [**3**] that in a real line arrangement, if a line crosses a multiple intersection point from one side to its other side (see Figure 2), the fundamental groups remain unchanged (see [**3**, Theorem 4.13] for this property of the action $\stackrel{\triangle}{=}$).

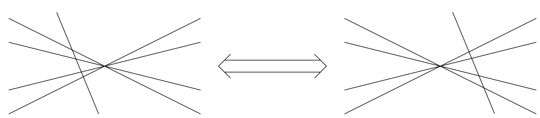

FIGURE 2. A line crosses a multiple intersection point.

By this argument, we can start with any transversal line L as "the line at infinity". Then, by using this property repeatedly, we can push this line over all the intersection points of the arrangement $\mathcal{L}$, without changing the corresponding fundamental group. This process will be terminated when all the intersection points of L with $\mathcal{L}$ are placed to the left of all the intersection points of $\mathcal{L}$, and this is the reduced case.

By this remark, we continue the proof of Theorem 1.3 for the reduced case, where all the intersection points of L with $\mathcal{L}$ are placed to the left of all the intersection points of $\mathcal{L}$. We compute presentations for $\pi_1(\mathbb{CP}^2 - \mathcal{L})$ and $\pi_1(\mathbb{CP}^2 - (\mathcal{L} \cup L))$ by braid monodromy techniques (the Moishezon-Teicher algorithm) and the van Kampen Theorem. The original techniques are introduced in [**6**] and [**5**] respectively. Shorter presentations of these techniques can be found in [**2**] and [**3**].

We first have to compute the lists of Lefschetz pairs of the arrangements, which are the pairs of indices of the components intersected at the intersection point, where we numerate the components locally near the point (see [**2**]). Since $\mathcal{L}$ is an arbitrary real line arrangement, its list of Lefschetz pairs is

$$([a_1, b_1], \ldots, [a_k, b_k]),$$

where k is the number of intersection points in $\mathcal{L}$. If we assume that the additional line L crosses all the lines of $\mathcal{L}$ transversally to the left of all the intersection points in $\mathcal{L}$, the list of Lefschetz pairs of $\mathcal{L} \cup L$ is

$$([a_1, b_1], \dots, [a_k, b_k], [n, n+1], \dots, [1, 2])$$

By the braid monodromy techniques and the van Kampen Theorem, the group $\pi_1(\mathbb{CP}^2 - \mathcal{L})$ has n generators $\{x_1, \dots, x_n\}$ and $\pi_1(\mathbb{CP}^2 - (\mathcal{L} \cup L))$ has $n+1$ generators $\{x_1, \dots, x_n, x_{n+1}\}$. Moreover, the first k relations of $\pi_1(\mathbb{CP}^2 - (\mathcal{L} \cup L))$ are equal to the k relations of $\pi_1(\mathbb{CP}^2 - \mathcal{L})$. Let us denote this set of relations by $\mathcal{R}$.

Now, we have to find out the relations induced by the n intersection points of the line L with the arrangement $\mathcal{L}$. Moreover, we have to add at last the appropriate projective relations.

First, we compute the relations induced by the n intersection points of the line L with the arrangement $\mathcal{L}$. The Lefschetz pair of the $(k+1)$th point (which is the first intersection point of L with $\mathcal{L}$) is $[n, n+1]$. Hence, its corresponding initial skeleton is shown in Figure 3.

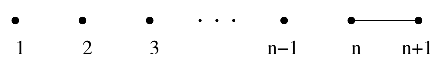

FIGURE 3. Initial skeleton of the $(k+1)$th point

Since the list of pairs $[a_1, b_1], \dots, [a_k, b_k]$ is induced by a line arrangement, it is easy to see that:

$$\Delta\langle a_k, b_k\rangle \cdots \Delta\langle a_1, b_1\rangle = \Delta\langle 1, n\rangle,$$

where $\Delta\langle 1, n\rangle$ is the generalized half-twist on the segment $[1, n]$.

Therefore, applying this braid on the initial skeleton yields the resulting skeleton for this point which is presented in Figure 4.

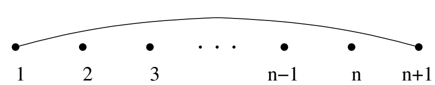

FIGURE 4. Final skeleton of the $(k+1)$th point

By the van Kampen Theorem, the corresponding relation is:

$$[x_n x_{n-1} \cdots x_2 x_1 x_2^{-1} \cdots x_n^{-1}, x_{n+1}] = 1$$

Let P_i be the ith intersection point, where $k+2 \leqslant i \leqslant k+n$. Its corresponding Lefschetz pair is $[n-(i-k)+1, n-(i-k)+2]$, and hence its initial skeleton is shown in Figure 5.

Now, we first have to apply on it the following braid:

$$\Delta\langle n-(i-k)+2, n-(i-k)+3\rangle \cdots \Delta\langle n, n+1\rangle$$

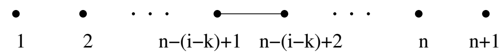

FIGURE 5. Initial skeleton of the ith point

and afterwards we have to apply on the resulting skeleton $\Delta\langle 1, n\rangle$, which equals to

$$\Delta\langle a_k, b_k\rangle \cdots \Delta\langle a_1, b_1\rangle$$

as before. Hence, Figure 6 presents the resulting skeleton for the ith point.

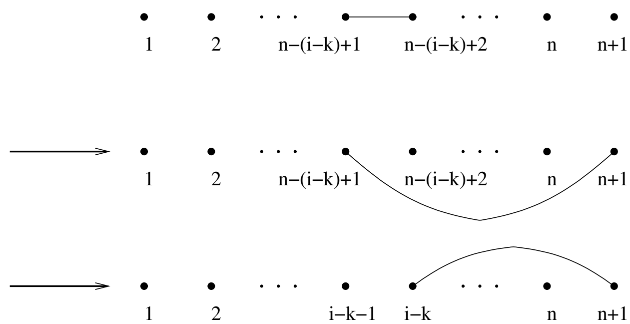

FIGURE 6. Computing the final skeleton of the ith point

Again, by the van Kampen Theorem, the corresponding relation is:

$$[x_n x_{n-1} \cdots x_{i-k+1} x_{i-k} x_{i-k+1}^{-1} \cdots x_n^{-1}, x_{n+1}] = 1$$

To summarize, we get that the set of relations induced by the intersection points of the additional line L is:

$$\{[x_n x_{n-1} \cdots x_{i-k+1} x_{i-k} x_{i-k+1}^{-1} \cdots x_n^{-1}, x_{n+1}] = 1 \mid 1 \leqslant i-k \leqslant n\}$$

One can easily see, by a sequence of substitutions, that actually this set of relations is equal to the following set:

$$\{[x_i, x_{n+1}] = 1 \mid 1 \leqslant i \leqslant n\}$$

Hence, we have the following presentations:

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) = \langle x_1, \dots, x_n \mid \mathcal{R}\ ;\ x_n x_{n-1} \cdots x_1 = 1\rangle$$

$$\pi_1(\mathbb{CP}^2 - (\mathcal{L} \cup L)) = \left\langle x_1, \dots, x_n, x_{n+1} \;\middle|\; \begin{array}{c} \mathcal{R}\ ;\ [x_i, x_{n+1}] = 1\ ,\ 1 \leqslant i \leqslant n\ ; \\ x_{n+1} x_n \cdots x_1 = 1 \end{array} \right\rangle$$

It remains to show that these presentations imply that:

$$\pi_1(\mathbb{CP}^2 - (\mathcal{L} \cup L)) = \mathbb{Z} \oplus \pi_1(\mathbb{CP}^2 - \mathcal{L})$$

Denote by $\mathcal{R}(x_1 \Leftarrow w)$ the set of relations $\mathcal{R}$ after we substitute anywhere the generator x_1 by an element w in the corresponding group.