Séminaires & Congrès
10, 2005, p. 71–86

ON THE PICARD GROUP FOR NON-COMPLETE ALGEBRAIC VARIETIES

by

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Abstract. — In this paper we show some relations between the topology of a complex algebraic variety and its algebraic or analytic Picard group. Some of our results involve the subgroup of the Picard group whose elements have a trivial Chern class and the Néron-Severi group, quotient of the Picard group by this subgroup. We are also led to give results concerning their relations with the topology of the complex algebraic variety.

***Résumé* (Sur le groupe de Picard des variétés algébriques non complètes).** — Dans cet article, nous montrons quelques relations entre la topologie d'une variété algébrique complexe et son groupe de Picard algébrique ou analytique. Certains de nos résultats concernent le sous-groupe du groupe de Picard dont les éléments ont une classe de Chern triviale et le groupe de Néron-Severi, quotient du groupe de Picard par ce sous-groupe. Nous obtenons aussi des résultats sur leurs relations avec la topologie de la variété algébrique complexe.

1. Statements

Let X be a complex algebraic variety, *i.e.* a (sc. separated) integral (*i.e.* irreducible and reduced) scheme of finite type over $\operatorname{Spec}\mathbb{C}$. Then we have a corresponding complex space X^{an}. The notion of the Picard group exists in the category of complex algebraic varieties and in the category of complex spaces, since both algebraic varieties and complex spaces are locally ringed spaces. Recall that, for a locally ringed space, the Picard group is the group of isomorphism classes of invertible sheaves. For algebraic varieties it coincides with the Cartier divisor class group [**H**] II 6.15.

If X is complete, *i.e.* X^{an} is compact, both Picard groups are isomorphic to each other by the GAGA principle: $\operatorname{Pic} X \simeq \operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}}$. If X is projective, this is a classical result of Serre [**S**]; for the general case see [**G2**] XII Th. 4.4. This is no longer true in general if X is not complete. This fact will be an easy consequence of Corollary

2000 *Mathematics Subject Classification.* — Primary 14 C 22; Secondary 14 C 30, 14 C 20, 32 J 25.
Key words and phrases. — Picard group, Hodge theory, Néron-Severi group.

1.3 below. A more interesting example is due to Serre, *cf.* [**H**] Appendix B 2.0.1; we thank the referee for drawing our attention to it: there are non-singular surfaces X_1 and X_2 such that $X_1^{\mathrm{an}} \simeq X_2^{\mathrm{an}}$ and $\operatorname{Pic} X_1 \not\simeq \operatorname{Pic} X_2$. So $\operatorname{Pic}_{(\mathrm{an})} X_1^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} X_2^{\mathrm{an}}$, X_1 is not isomorphic to X_2, and we cannot have $\operatorname{Pic} X_j \simeq \operatorname{Pic}_{(\mathrm{an})} X_j^{\mathrm{an}}$, $j = 1, 2$.

We will concentrate here upon the case where X is non-singular. Remember that we have a canonical mixed Hodge structure on the cohomology groups of X^{an} [**D1**].

As usual, if $(\mathbf{H}, F, W)$ is a mixed Hodge structure on $\mathbf{H}$, F is the Hodge filtration $\cdots \supset F^n\mathbf{H}_{\mathbb{C}} \supset F^{n+1}\mathbf{H}_{\mathbb{C}} \supset \ldots$ on $\mathbf{H}_{\mathbb{C}} := \mathbf{H} \otimes \mathbb{C}$ and W is the weight filtration on $\mathbf{H}_{\mathbb{Q}} := \mathbf{H} \otimes \mathbb{Q}$

$$\cdots \subset W_k\mathbf{H}_{\mathbb{Q}} \subset W_{k+1}\mathbf{H}_{\mathbb{Q}} \subset \ldots$$

We write

$$\mathrm{Gr}_\ell^W \mathbf{H}_{\mathbb{Q}} = W_\ell\mathbf{H}_{\mathbb{Q}}/W_{\ell-1}\mathbf{H}_{\mathbb{Q}} \text{ and } \mathrm{Gr}_F^i \mathbf{H}_{\mathbb{C}} = F^i\mathbf{H}_{\mathbb{C}}/F^{i+1}\mathbf{H}_{\mathbb{C}}.$$

Recall also that the Hodge filtration induces a filtration on each $\mathrm{Gr}_\ell^W \mathbf{H}_{\mathbb{C}}$.

In contrast to the approach of A. Grothendieck [**G1**] we apply transcendental methods which lead to results involving transversality conditions.

First let us study the question whether $\operatorname{Pic} X$ is trivial:

1.1. Theorem. — *Let X be a non-singular complex algebraic variety, assume that* $\mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$, $\mathrm{Gr}_F^1 \mathrm{Gr}_2^W H^2(X^{\mathrm{an}}; \mathbb{C}) = 0$ *and* $H^2(X^{\mathrm{an}}; \mathbb{Z})$ *is torsion free. Then* $\operatorname{Pic} X = 0$, *i.e. every divisor on X is a principal divisor.*

Note that there is no difference between Weil and Cartier divisors here because X is supposed to be non-singular (see [**H**] Chap. II 6.11.1 A). Of course, this theorem shows that it is sufficient to suppose that $H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$ and $H^2(X^{\mathrm{an}}; \mathbb{Z}) = 0$ to obtain $\operatorname{Pic} X = 0$. In particular, it is not possible to distinguish X from the affine space $\mathbb{A}_n = \mathbb{A}_n(\mathbb{C}) := \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ by the Picard group if X^{an} is contractible.

Conversely, we have:

1.2. Theorem. — *Let X be a non-singular complex algebraic variety and suppose* $\operatorname{Pic} X = 0$. *Then*

$$\mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0 \textit{ and } H^2(X^{\mathrm{an}}; \mathbb{Z}) \textit{ is torsion free.}$$

Then we also get the following easy consequence of both theorems:

1.3. Corollary. — *Let X be a non-complete non-singular irreducible complex curve, g the genus of its non-singular compactification $\overline{X}$. Then $g = 0$ if and only if the algebraic Picard group* $\operatorname{Pic} X$ *of X is trivial.*

Note, however, that the analytic Picard group $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}}$ is always trivial in the case of a non-complete irreducible complex curve.

Now let us turn to the Picard group in the case where it is non-trivial. In general, the structure of the Picard group can be quite complicated but we have a comparison theorem:

1.4. Theorem. — *Let $f : Y \to X$ be a morphism between non-singular complex algebraic varieties. Suppose that the induced map*

$$H^k(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^k(Y^{\mathrm{an}};\mathbb{Z})$$

is bijective for $k = 1, 2$. Then the natural map $\operatorname{Pic} X \to \operatorname{Pic} Y$ is bijective.

As a consequence, there is a theorem of Zariski-Lefschetz type, using a corresponding topological theorem [**HL**]. Let us state it in a slightly more general form, admitting singularities.

Now, X might be singular. Let $\operatorname{Cl} X$ be the Weil divisor class group of X and $\operatorname{Sing} X$ the singular locus of X.

1.5. Theorem. — *Let $\operatorname{Sing} X$ be of codimension $\geqslant 2$ in X and let $\overline{X}$ be a compactification of X to a projective variety embedded in $\mathbb{P}_m = \mathbb{P}_m(\mathbb{C})$. Let us fix a stratification of $\overline{X}$ such that X and $X \smallsetminus \operatorname{Sing} X$ are unions of strata. Let Z be a complete intersection in $\mathbb{P}_m$ which is non-singular along $\overline{X}$ and intersects all strata of $\overline{X}$ transversally in $\mathbb{P}_m$, and let $Y := X \cap Z$. Suppose $\dim Y \geqslant 3$. Then $\operatorname{Cl} X \simeq \operatorname{Cl} Y$. If X is affine, we have $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}}$, too.*

Note that Cl may be replaced by Pic if X is non-singular ([**H**] Chap. II 6.16).

1.6. Corollary. — *Suppose that X is a non-singular affine variety of dimension $\geqslant 3$ in $\mathbb{P}_m$. Then there is a linear subspace L of $\mathbb{P}_m$ such that $Y = X \cap L$ is non-singular, $\dim Y = 3$ and $\operatorname{Pic} X \simeq \operatorname{Pic} Y$, $\operatorname{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq \operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}}$.*

1.7. Corollary. — *Let Y be a non-singular closed subvariety of the affine space $\mathbb{A}_m$, $\dim Y \geqslant 3$. Assume that the closure $\overline{Y}$ in $\mathbb{P}_m$ is a non-singular complete intersection which is transversal to $\mathbb{P}_m \smallsetminus \mathbb{A}_m$. Then $\operatorname{Pic} Y = 1$, $\operatorname{Pic}_{(\mathrm{an})} Y^{\mathrm{an}} = 1$.*

In fact, this last corollary is a simultaneous consequence of Theorem 1.1 and 1.5, which justifies to treat both theorems here at the same time.

We are grateful to U. Jannsen for drawing our attention to related developments in the theory of mixed motives [**J**].

2. Proofs of Theorems 1.1 and 1.2

Let X be a smooth complex algebraic variety of dimension n. Recall that we can attach to each invertible sheaf on X its first Chern class. This gives a homomorphism $\alpha : \operatorname{Pic} X \to H^2(X^{\mathrm{an}};\mathbb{Z})$. Let $\operatorname{Pic}^0 X$ be the kernel.

Since X is separated there is a compactification $\overline{X}$ by Nagata [**N**]. Since X is smooth we can obtain by Hironaka [**Hi**] that $\overline{X}$ is smooth and that $\overline{X} \smallsetminus X$ is a divisor with normal crossings $D = D_1 \cup \cdots \cup D_r$, where the components $D_1, \ldots, D_r$ are smooth.

Recall that, for all k, $W_{k-1}H^k(X^{\mathrm{an}};\mathbb{Q}) = 0$, because X is non-singular, see [**D1**] 3.2.15.

2.1. Lemma. — *The canonical mapping $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \to H^1(X^{\mathrm{an}};\mathbb{Q})$ is injective, the image is $W_1H^1(X^{\mathrm{an}};\mathbb{Q}) \simeq \mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})$.*

Proof. — Let us look at the exact sequence

$$H^1(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Q}) \longrightarrow H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \longrightarrow H^1(X^{\mathrm{an}};\mathbb{Q})$$

By Lefschetz duality $H^1(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}};\mathbb{Q})$ is dual to the vector space $H^{2n-1}(D^{\mathrm{an}};\mathbb{Q})$ which vanishes because $\dim D = n-1$. This proves the injectivity.

On the other hand, the image of $H^1(\overline{X}^{\mathrm{an}};\mathbb{Q}) \to H^1(X^{\mathrm{an}};\mathbb{Q})$ is $W_1H^1(X^{\mathrm{an}};\mathbb{Q}) \simeq \mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q})$ by [**D1**] p. 39, Cor. 3.2.17. □

2.2. Proposition. — *The following conditions are equivalent:*

a) $\mathrm{Pic}\, X$ *is a finitely generated group,*
b) $\mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q}) = 0$,
c) $\mathrm{Pic}^0 X = 0$.

Proof. — Let us first consider the case where X is complete. Since X is also supposed to be smooth, the mixed Hodge structure on $H^1(X^{\mathrm{an}};\mathbb{Q})$ is pure of weight 1, so

$$H^1(X^{\mathrm{an}};\mathbb{Q}) = \mathrm{Gr}_1^W H^1(X^{\mathrm{an}};\mathbb{Q}).$$

Therefore b) is equivalent to the condition $b^1(X^{\mathrm{an}}) = 0$, where b^1 denotes the first Betti number. Now the latter can be expressed by the Hodge numbers: $b^1(X^{\mathrm{an}}) = h^{01}(X^{\mathrm{an}}) + h^{10}(X^{\mathrm{an}}) = 2h^{01}(X^{\mathrm{an}})$. Note that X^{an} need not be a Kähler manifold, since X might not be projective. Anyhow X is algebraic, and we have $h^{pq}(X^{\mathrm{an}}) = \dim_{\mathbb{C}} H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$ because of the definition of the Hodge filtration in general, see [**D1**] (2.2.3) et (2.3.7).

So b) is equivalent to the condition that $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) = 0$.

Now the exponential sequence leads to the following exact sequence:

$$H^1(X^{\mathrm{an}};\mathbb{Z}) \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow \mathrm{Pic}\, X \xrightarrow{\alpha} H^2(X^{\mathrm{an}};\mathbb{Z}).$$

Here we use the fact that $\mathrm{Pic}\, X \simeq \mathrm{Pic}_{(\mathrm{an})} X^{\mathrm{an}} \simeq H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}})$ by GAGA, because X is complete. Now $H^1(X^{\mathrm{an}};\mathbb{Z})$ and $H^2(X^{\mathrm{an}};\mathbb{Z})$ are finitely generated abelian groups, *i.e.* Noetherian $\mathbb{Z}$-modules. In particular, $\mathrm{Pic}\, X/\mathrm{Pic}^0 X$ is finitely generated.

a) $\Rightarrow$ b): By the exact sequence above, if $\mathrm{Pic}\, X$ is finitely generated, the cohomology group $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is also a finitely generated group. But, since we consider a complex vector space, it is a finitely generated group if and only if it is trivial.

b) $\Rightarrow$ c): follows from the surjectivity of $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to \mathrm{Pic}^0 X$.

c) $\Rightarrow$ *a*) As said before, $\mathrm{Pic}\, X/\mathrm{Pic}^0 X$ is finitely generated.

This finishes the special case where X is complete.

Now let us turn to the general case. Since X and $\overline{X}$ are smooth we can replace the Picard group by the Weil divisor class group, so we have an exact sequence of the form

$$\mathbb{Z}^r \longrightarrow \operatorname{Pic}\overline{X} \longrightarrow \operatorname{Pic} X \longrightarrow 0$$

see [**H**] II Prop. 6.5, p. 133 in the case $r = 1$.

a) $\Rightarrow$ b): Since $\operatorname{Pic} X$ is finitely generated, the same holds for $\operatorname{Pic}\overline{X}$. By the first case, $H^1(\overline{X}^{\mathrm{an}}, \mathbb{Q}) = 0$. Now Lemma 2.1 yields $\mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$.

b) $\Rightarrow$ c): By Lemma 2.1, $H^1(\overline{X}^{\mathrm{an}}; \mathbb{Q}) \simeq \mathrm{Gr}_1^W H^1(X^{\mathrm{an}}; \mathbb{Q}) = 0$, so

$$\operatorname{Pic}^0 \overline{X} = 0$$

by the first case applied to $\overline{X}$ which is complete. Now, let us consider the commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbb{Z}^r \arrow[r] \arrow[d, "\wr"'] & \operatorname{Pic}\overline{X} \arrow[r] \arrow[d, "\overline{\alpha}"'] & \operatorname{Pic} X \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[d] \\
H^2(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}}; \mathbb{Z}) \arrow[r] & H^2(\overline{X}^{\mathrm{an}}; \mathbb{Z}) \arrow[r] & H^2(X^{\mathrm{an}}; \mathbb{Z}) \arrow[r] & H^3(\overline{X}^{\mathrm{an}}, X^{\mathrm{an}}; \mathbb{Z})
\end{tikzcd}$$

Here we were allowed to put the right hand vertical arrow by a diagram chase. Since $\operatorname{Pic}^0 \overline{X} = 0$, we know that $\overline{\alpha}$ is injective. The five lemma shows therefore that α is also injective. This means that $\operatorname{Pic}^0 X = 0$.

c) $\Rightarrow$ a): This follows from the fact that $\operatorname{Pic} X / \operatorname{Pic}^0 X$ is finitely generated. $\square$

Proof of Theorem 1.1. — By Proposition 2.2 we have that $\operatorname{Pic}^0 X = 0$, so the natural mapping $\operatorname{Pic} X \to H^2(X^{\mathrm{an}}; \mathbb{Z})$ is injective.

If X is complete, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Pic} X \longrightarrow H^2(X^{\mathrm{an}}; \mathbb{Z}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$$

We can factorize the last map through $H^2(X^{\mathrm{an}}; \mathbb{C})$. The image P of $\operatorname{Pic} X$ in $H^2(X^{\mathrm{an}}; \mathbb{C})$ is obviously contained in the kernel of the map

$$H^2(X^{\mathrm{an}}; \mathbb{C}) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \simeq \mathrm{Gr}_F^0 H^2(X^{\mathrm{an}}; \mathbb{C}).$$

This kernel is

$$U := F^1 H^2(X^{\mathrm{an}}; \mathbb{C})$$

because $H^2(X^{\mathrm{an}}; \mathbb{C}) = F^0(H^2(X^{\mathrm{an}}; \mathbb{C}))$ and

$$\mathrm{Gr}_F^0(H^2(X^{\mathrm{an}}; \mathbb{C})) = F^0(H^2(X^{\mathrm{an}}; \mathbb{C}))/F^1(H^2(X^{\mathrm{an}}; \mathbb{C})).$$

Since $\operatorname{Pic} X$ injects in $H^2(X^{\mathrm{an}}; \mathbb{Z})$, P is also invariant under conjugation, so P is contained in $U \cap \overline{U}$.

We observe that, by definition of the Hodge structure (see [**D1**] (B) of (2.2.1)) we have

$$U \cap \overline{U} \simeq \mathrm{Gr}_F^1 H^2(X^{\mathrm{an}}; \mathbb{C})$$