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THREE KEY THEOREMS ON INFINITELY NEAR SINGULARITIES

by

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***Abstract.* —** The notion of infinitely near singular points is classical and well understood for plane curves. We generalize the notion to higher dimensions and to develop a general theory, in terms of *idealistic exponents* and certain graded algebras associated with them. We then gain a refined generalization of the classical notion of first characteristic exponents. On the level of technical base in the higher dimensional theory, there are some powerful tools, referred to as *Three Key Theorems*, which are namely *Differentiation Theorem*, *Numerical Exponent Theorem* and *Ambient Reduction Theorem*.

***Résumé* (Trois théorèmes-clefs sur les singularités infiniment proches). —** La notion de points singuliers infiniment proches est classique et bien comprise pour les courbes planes. On généralise cette notion aux plus grandes dimensions et on développe une théorie générale, en termes de d'*exposants idéalistes* et certaines algèbres graduées associées. Ainsi on obtient une généralisation raffinée de la notion classique des premiers exposants caractéristiques. Au niveau technique de base dans la théorie de dimension plus grande, on a des outils puissants, appelés les *Trois théorèmes-clefs*. Ce sont *le Théorème de différenciation*, *le Théorème de l'exposant numérique* et *le Théorème de réduction de l'espace ambiant*.

Introduction

The notion of infinitely near singular points is classical and well understood for plane curves. In order to generalize the notion to higher dimensions and to develop a general theory, we introduced the notion of *idealistic exponents* which, in the plane curve case, correspond to the first characteristic exponents. On the level of technical base in the higher dimensional theory, there are some powerful tools, referred to as

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Three Key Theorems, which are namely *Differentiation Theorem*, *Numerical Exponent Theorem* and *Ambient Reduction Theorem*. In this paper the three key theorems will be proven for singular data on an ambient regular scheme of finite type over any perfect field of any characteristics. In the proofs, the role played by differential operators will be ubiquitous and indispensable. The notion and basic properties of differential operators will be reviewed in the first chapter, in a manner that is purely algebraic and abstract. In the last two chapters, we state and prove the *Finite Presentation Theorem* as an application of the Key Theorems. The finite presentation is the first step and is believed by the author to be an important milestone in the development of a general theory of infinitely near singular points, giving an *algebraic presentation of finite type* to the total aggregate of all the trees of infinitely near singular points, geometrically diverse and intricate. The original proof of this theorem is contained in a paper which is going to be published in the Journal of the Korean Mathematical Society, but it is repeated here for the sake of emphasizing how important are the roles played by the key theorems. Technically in this work at least, the general theory of infinitely near singular points in higher dimensions heavily depends upon the use of partial differential operators. This approach is interesting in its own right, for instance as was shown by Jean Giraud in connection with the theory of *maximal contact*, [**3, 4**]. As as final comment, now that the algebraic presentation of finite type is known, the next charming project will be the study of structure theorems of the presentation algebras which contain rich information on the given singular data.

Notation. — Our terminal interest of this paper concerns with schemes of finite type over a perfect base field $\mathbf{k}$, which may have any characteristic. However, our interest beyond this paper will be about schemes of finite type over any excellent Dedeking domain, which will be denoted by $\Bbbk$. For examples, $\Bbbk$ could be any field or the ring of integers in any algebraic number field. From time to time, however, we choose to work on a more abstract and general scheme when possible and desirable. For instance, our schemes may be finite type over any noetherian ring, denoted by B. This B could be the completion of a local ring of a scheme.

1. Differential operators

For the sake of generality, let R be any commutative B-algebra, where B is a commutative ring, and we first define a left R-algebra by the action of the elements of R on the left:

$$\Omega^{(\mu)}_{R/B} = \big(R \otimes_B R\big)/D_R^{\mu+1}$$

where μ is any non-negative integer and D_R denotes the diagonal ideal in the tensor product, which means the kernel

$$D_R = \operatorname{Ker}\big(R \otimes_B R \longrightarrow R\big)$$

of the map induced by the multiplication law of R. We also have

$$D_R = \{\delta(f) \mid f \in R\} \subset R \otimes_B R, \quad \text{where } \delta(f) \text{ denotes } 1 \otimes f - f \otimes 1$$

The differential operators of orders $\leqslant \mu$ are defined to be the elements of the dual of $\Omega^{(\mu)}_{R/B}$. Namely, they are the elements of

$$\mathrm{Diff}^{(\mu)}_{R/B} = \mathrm{Hom}_R\big(\Omega^{(\mu)}_{R/B}, R\big)$$

We often identify elements of $\mathrm{Diff}^{(\mu)}_{R/B}$ with R-homomorphism from $R \otimes_B R$ to R via the natural homomorphism $R \otimes_B R \to \Omega^{(\mu)}_{R/B}$. In this sense, we have canonical inclusions

$$\mathrm{Diff}^{(\mu)}_{R/B} \subset \mathrm{Diff}^{(\nu)}_{R/B} \quad \text{whenever } \mu \leqslant \nu$$

Accordingly we sometimes write

$$\mathrm{Diff}_{R/B} \quad \text{for} \quad \bigcup_{\forall\, \nu \geqslant 0} \mathrm{Diff}^{(\nu)}_{R/B}$$

Furthermore, an element $\partial \in \mathrm{Diff}^{(\mu)}_{R/B}$ acts on elements of R by

$$f \in R \longmapsto \partial(1 \otimes f) \in R$$

in which sense ∂ will be often viewed as an element of $\mathrm{Hom}_B(R, R)$. It is B-linear but hardly R-linear. When a B-subalgebra S of R is given, we have a natural epimorphism $R \otimes_B R \to R \otimes_S R$ which maps the diagonal ideal of the former to that of the latter. Hence we get epimorphisms $\Omega^{(\mu)}_{R/B} \to \Omega^{(\mu)}_{R/S}, \forall\, \mu$, so that we have canonical monomorphisms $\mathrm{Diff}^{(\mu)}_{R/S} \to \mathrm{Diff}^{(\mu)}_{R/B}$. In this sense, we will often view $\mathrm{Diff}^{(\mu)}_{R/S}$ as an R-submodule of $\mathrm{Diff}^{(\mu)}_{R/B}$.

Lemma 1.1. — *Let T be any multiplicative subset of R. Then, viewing $\Omega^{(\mu)}_{R/B}$ and $\mathrm{Diff}^{(\mu)}_{R/B}$ as left R-modules, we have the following compatibility with localizations by T:*

$$\Omega^{(\mu)}_{(T^{-1}R)/B} = T^{-1}\Omega^{(\mu)}_{R/B}$$

and if $\Omega^{(\mu)}_{R/B}$ is finitely generated as an R-module then

$$\mathrm{Diff}^{(\mu)}_{(T^{-1}R)/B} = T^{-1}\,\mathrm{Diff}^{(\mu)}_{R/B}$$

Proof. — For every $t \in T$, we have $f \otimes 1 + \delta(t) = 1 \otimes t$. Here the multiplication by $f \otimes 1$ on $T^{-1}\Omega^{(\mu)}_{R/B}$ is invertible while that by $\delta(t)$ is nilpotent. Hence the multiplication by $1 \otimes t$ is invertible. Namely $(1 \otimes T)^{-1}\big(T^{-1}\Omega^{(\mu)}_{R/B}\big) = T^{-1}\Omega^{(\mu)}_{R/B}$. Moreover, we have

$$\Omega^{(\mu)}_{(T^{-1}R)/B} = (1 \otimes T)^{-1}(T \otimes 1)^{-1}\Omega^{(\mu)}_{R/B} = (1 \otimes T)^{-1}\big(T^{-1}\Omega^{(\mu)}_{R/B}\big)$$

which proves the first assertion of the lemma. The second assertion is by the commutativity of Hom and localizations for finitely generated modules. □

Lemma 1.2. — *Let* $P = B[z]$ *be the polynomial ring of independent variables* $z = (z_1, \dots, z_N)$. *Then*

$$\Omega^{(m)}_{P/B} = P[\delta(z)]/\big(\delta(z)\big)^{m+1}P[\delta(z)]$$

which is freely generated as P*-module by the images of the monomials of degrees* $\leqslant m$ *in the independent variables* $\delta(z)$ *over* P. *Consequently,*

$$\mathrm{Diff}^{(m)}_{P/B} = \sum_{\substack{\alpha\in\mathbb{Z}^N\\ |\alpha|\leqslant m}} P\partial_\alpha$$

where

$$\partial_\alpha z^\beta = \begin{cases} \binom{\beta}{\alpha} z^{\beta-\alpha} & \text{if } \beta \in \alpha + \mathbb{Z}^N_0 \\ 0 & \text{if otherwise} \end{cases}$$

Moreover, for $\zeta \in \mathrm{Spec}(P)$ *and* $A = P/\zeta$, *we have*

$$\Omega^{(m)}_{A/B} = \Omega^{(m)}_{P/B}\Big/\Big(\zeta\Omega^{(m)}_{P/B} + P\delta(\zeta)\Big)$$

and therefore $\mathrm{Diff}^{(m)}_{A/B}$ *is a finite* A*-module.*

Proof. — In fact, in $P \otimes_B P$ as left P-algebra, we may identify $z \otimes 1$ with z itself and therefore $P \otimes_B P$ with $P[1 \otimes z] = P[\delta(z)]$, where $\delta(z) = \big(\delta(z_1), \dots, \delta(z_N)\big)$. Hence

$$\Omega^{(m)}_{P/B} = P[\delta(z)]/\big(\delta(z)\big)^{m+1}P[\delta(z)]$$

which has the asserted property. Hence, its dual $\mathrm{Diff}^{(m)}_{P/B}$ has also the asserted property. As for the assertion on $\Omega^{(m)}_{A/B}$, it is enough to see that

$$\big((\zeta \otimes 1) + (1 \otimes \zeta)\big)P \otimes_B P = \zeta(P \otimes_B P) + P\delta(\zeta) \qquad \square$$

Now, in the case of an affine scheme $Z = \mathrm{Spec}(A)$ where A is finitely generated as B-algebra and B is noetherian, we define $\Omega^{(\mu)}_{Z/B}$ to be the *coherent* $\mathcal{O}_Z$*-algebra* which corresponds to the finite A-algebra $\Omega^{(\mu)}_{A/B}$. Similarly, we define $\mathrm{Diff}^{(\mu)}_Z$ to be the *coherent* $\mathcal{O}_Z$*-module* which correspond to the finite A-module $\mathrm{Diff}^{(\mu)}_{A/B}$. The finiteness and coherency are due to Lemma 1.2. Since the definition of these A-modules commutes with localizations of A by Lemma 1.1, the definitions of $\Omega^{(\mu)}_{Z/B}$ and $\mathrm{Diff}^{(\mu)}_{Z/B}$ are naturally globalized for any scheme Z, not necessarily affine, of finite type over B. We call $\Omega^{(\mu)}_{Z/B}$ the $\mathcal{O}_Z$*-algebra of* μ*-jets of* Z *over* B and $\mathrm{Diff}^{(\mu)}_{Z/B}$ the $\mathcal{O}_Z$*-module of differential operators of orders* $\leqslant \mu$ *of* Z *over* B. We sometimes write $\mathrm{Diff}^{(\mu)}_Z$ for $\mathrm{Diff}^{(\mu)}_{Z/B}$ if the reference to B is clear from the context.

Back to a general commutative B-algebra R and $Z = \mathrm{Spec}(R)$, we will prove two useful lemmas on $\mathrm{Diff}^{(\mu)}_{Z/B}$, the first one is about *compositions* and the second about *commutators* of differential operators of R over B. The third lemma is a consequence of the two which we need later. In the proofs of the first two lemmas, we will follow the following chain of R-homomorphisms for a pair of differential operators ∂ and ∂':

$$R \otimes_B R \xrightarrow{(1,3)} R \otimes_B R \otimes_B R \xrightarrow{(1,\partial)} R \otimes_B R \xrightarrow{\partial'} R \tag{1.1}$$

where $(1,3) : f \otimes g \mapsto f \otimes 1 \otimes g$ and $(1,\partial) : f \otimes g \otimes h \mapsto f \otimes \partial(f \otimes g)$. Here $\partial \in \mathrm{Diff}^{(\mu)}_{R/B}$ is viewed as an R-homomorphism from $R \otimes_B R$ to R through the natural surjection $R \otimes_B R \to \Omega^{(\mu)}_{R/B}$. Likewise for ∂'. It should be noted that for every $f \in R$ the end image of $1 \otimes f$ by the above (1.1) is exactly $(\partial' \circ \partial)(f)$ in the sense of composition $\partial' \circ \partial$ of the two differential operators as being viewed as endomorphisms of R. When there is no ambiguity, we sometimes write $\partial'\partial$ for $\partial' \circ \partial$.

Lemma 1.3. — *Viewing $\partial \in \mathrm{Diff}^{(\mu)}_{R/B}$ and $\partial' \in \mathrm{Diff}^{(\mu')}_{R/B}$ as endomorphisms of R, we have the composition $\partial' \circ \partial$ belong to $\mathrm{Diff}^{(\mu+\mu')}_{R/B}$. Namely we have a natural homomorphism $\mathrm{Diff}^{(\mu)}_{R/B} \times \mathrm{Diff}^{(\mu')}_{R/B} \to \mathrm{Diff}^{(\mu+\mu')}_{R/B}$.*

Proof. — What we want is that if γ denotes the composition of the chain of homomorphisms of (1.1) then $\gamma(D_R^{\mu+\mu'+1}) = 0$. Define $(i,j) : R \otimes_B R \to R \otimes_B R \otimes_B R$ for $1 \leqslant i < j \leqslant 3$ in the same way as the above $(1,3)$ and let $D_{i,j} = (i,j)(D_R)$. Then we have $D_{1,3} \subset D_{1,2} + D_{2,3}$ because

$$1 \otimes 1 \otimes f - f \otimes 1 \otimes 1 = (1 \otimes f \otimes 1 - f \otimes 1 \otimes 1) + (1 \otimes 1 \otimes f - 1 \otimes f \otimes 1)$$

We then obtain

$$D_{1,3}^{\mu+\mu'+1} \subset (D_{1,2} + D_{2,3})^{\mu+\mu'+1} \subset D_{1,2}^{\mu'+1} + D_{2,3}^{\mu+1}$$

Since $\partial'(D_R^{\mu'+1}) = \partial(D_R^{\mu+1}) = 0$, there follows $\gamma(D_R^{\mu+\mu'+1}) = 0$. □

Lemma 1.4. — *For ∂ and ∂' as above, we have the following inclusion of the commutator:*

$$[\partial', \partial] = \partial' \circ \partial - \partial \circ \partial' \in \mathrm{Diff}^{(\mu'+\mu-1)}_{R/B}$$

Proof. — Pick any system of $\mu' + \mu$ elements $g_j \in R$. Let γ be the composition of (1.1) as before, and let γ' be the similar composition when ∂ and ∂' are exchanged in (1.1). It is then enough to prove that

$$\gamma\Big(\prod_{j=1}^{\mu'+\mu} \delta(g_j)\Big) = \gamma'\Big(\prod_{j=1}^{\mu'+\mu} \delta(g_j)\Big) \tag{1.2}$$

Now, writing $\delta_{i,j} = (i,j) \circ \delta$, we obtain

$$\prod_{j=1}^{\mu'+\mu} \delta_{1,3}(g_j) \equiv \sum_{\substack{I_1 \cup I_2 = [1,\mu'+\mu] \\ I_1 \cap I_2 = \varnothing \\ |I_1| = \mu', |I_2| = \mu}} \Big(\prod_{k \in I_1} \delta_{1,2}(g_k)\Big)\Big(\prod_{l \in I_2} \delta_{2,3}(g_l)\Big) \qquad \text{modulo } D_{1,2}^{\mu'+1} + D_{2.3}^{\mu+1}$$