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PROPERTIES OF LAMÉ OPERATORS WITH FINITE MONODROMY

by

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Abstract. — This survey paper contains recent developments in the study of Lamé operators having finite monodromy group. We present the approach based on the pull-back theory of Klein, that allowed the description of the projective monodromy groups by Baldassarri ([**Bal81**]), as well as the connection with Grothendieck's theory of dessins d'enfants, that leads to explicit properties and formulae. The results of Beukers and van der Waall ([**BvdW04**]) concerning the full monodromy group are also presented. The last section describes the Lamé operators L_1 with finite monodromy in terms of the values of the Weierstraß zeta function corresponding to the elliptic curve associated to L_1, as well as the connection with the modular forms.

***Résumé* (Propriétés des opérateurs de Lamé à monodromie finie).** — Cet article présente quelques développements récents dans l'étude des opérateurs de Lamé à monodromie finie. On décrit l'approche basée sur la théorie des pull-back développée par Klein et utilisée par Baldassarri ([**Bal81**]) pour décrire la monodromie projective. On fait ensuite le lien avec la théorie des dessins d'enfants de Grothendieck, qui amène à des descriptions et à des formules explicites. On revient également sur les résultats de Beukers and van der Waall ([**BvdW04**]) concernant la monodromie. La dernière partie est consacrée à l'étude des opérateurs de Lamé L_1 avec monodromie finie en termes des valeurs de la fonction zéta de Weierstraß correspondant à la courbe elliptique attachée à L_1 et au lien avec les formes modulaires.

1. Introduction

Let C be an algebraic curve defined over $\mathbb{C}$ (smooth, projective and geometrically connected), or, equivalently a compact Riemann surface. We denote $K = K(C)$ the function field of C.

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Let D be a nontrivial derivation on K over $\mathbb{C}$ and

$$L = D^m + p_1 D^{m-1} + \cdots + p_{m-1} D + p_m \tag{1.1}$$

be a linear differential operator of order m on C, where $p_i \in K$ for $i \in \{1, \dots, m\}$. If $P \in C$ corresponds to the valuation v_P of K and t is a local parameter at P, then locally

$$L = q\left(\frac{d^m}{dt^m} + p_1' \frac{d^{m-1}}{dt^{m-1}} + \cdots + p_{m-1}' \frac{d}{dt} + p_m'\right) \tag{1.2}$$

where $q, p_i' \in K$, $i \in \{1, \dots, m\}$. The point P is a regular point for L if $v_P(p_i') \geq 0$ for all $i \in \{1, \dots, m\}$ and a singular point otherwise. Obviously the set S of singular points of L is finite, let $S = \{P_1, P_2, \dots, P_r\}$. If $v_P(p_i') \geq -i$ then the singular point P is called regular. At each regular point L has n independent solutions which are holomorphic. We shall suppose that all the singular points of the operators we are dealing with in this paper are regular and, moreover, if P is a regular singular point of an operator L as in (1.1) then L has m independent solutions at P of the form

$$u_i = t^{\alpha_i}(c_0 + c_1 t + \dots) \tag{1.3}$$

$i = 0, \dots, m$, with $\alpha_i \in \mathbb{Q}$. The rational numbers α_i are called the exponents of L at P and they are the roots of a polynomial equation of degree m, the indicial equation. Under these assumptions, if all the exponents are distinct, but differ only by integers, then every solution $y(t)$ is either holomorphic, or can be made so locally around P after a transformation $y = t^\rho y^*$ (see Poole [**Poo60**]).

If $P \in C \setminus S$, analytic continuation of the functions in a basis of solutions in P yields to the monodromy representation

$$\pi_1(C \setminus S) \to GL(m, \mathbb{C}) \tag{1.4}$$

For various points P and different basis of solutions, these representations are conjugated to each other. The image is called the monodromy group of the operator L. It is a subgroup of $GL(m, \mathbb{C})$, well-defined up to conjugation. The monodromy group is in general a subgroup of the differential Galois group attached to the operator L. If the singular points of L are regular, then the differential Galois group and the Zariski closure of the monodromy group coincide.

It is well known that, in general, a differential operator L is parameterised by the set of singular points S, the set E of values of the mr exponents and $v_{g,m}(r)$ accessory parameters: for example (see Ince [**Inc44**] or Dwork [**Dwo90**])

$$v_{0,m}(r) = (m-1)[m(r-2) - 2]/2. \tag{1.5}$$

Let B be the set of the accessory parameters.

For the rest of this paper we shall consider only second order differential operators. If τ is the ratio of two functions in a basis of the set of solutions of L at an arbitrary point $P \in C \setminus S$, the analytic continuation of τ along the paths in $\pi_1(P, C \setminus S)$ yields to a map

$$\pi_1(P, C \setminus S) \to PGL(2, \mathbb{C}) \tag{1.6}$$

The image of this map is called the projective monodromy group of the operator L. Its conjugation class does not depend on P, nor on τ.

If α_1, α_2 are the exponents of the operator L at a point $P \in C$, let $\Delta_{P,L} = |\alpha_1 - \alpha_2|$ be the *exponent difference* of L at P and $\Delta_L = \sum_{P \in \mathbb{P}^1} (\Delta_{P,L} - 1)$. Hereafter, a singular point where the exponent difference is an integer greater than 2 is called a *quasi-apparent singularity.* As in [**BvdW04**], a second order operator L is called *pure* if it has no quasi-apparent singularity.

Definition 1.1. — The couples (C, L), (C', L') are called projectively equivalent if there exists an isomorphism $f : C \to C'$ such that L is a weak pull-back of L' via f.

In this situation, L and L' have the same projective monodromy group and the same exponent differences. Throughout this paper, an abstract operator will be an equivalence class of couples (C, L). Eventually, the curve C may not be mentioned explicitly, if no confusion is possible.

Let now $f : C \to C'$ be a non constant morphism of algebraic curves and L and L' be second order linear differential operators on C and C' respectively. We say that L is a weak pull-back of L' via f if $\tau' \circ f$ is a ratio of independent solutions of L, provided that τ' is a ratio of independent solutions of L'. As we are interested in studying the set of differential operators modulo the projective equivalence, we shall use freely in this paper the notation f^*L' for a weak pull-back of the operator L'. If $L = f^*L'$, it follows immediately that $\Delta_{P,L} = e_P \cdot \Delta_{f(P),L'}$ for any $P \in C$, where e_P is the ramification index of f at P. The Riemann-Hurwitz formula implies (see Baldassarri and Dwork [**BD79**], Lemma 1.5, or Baldassarri [**Bal80**])

$$\Delta_L + 2 - 2g(C) = \deg f \cdot (\Delta_{L'} + 2 - 2g(C')). \tag{1.7}$$

2. Second order differential operators with algebraic solutions

The problem we are interested in is the following: which are the conditions that one has to impose on the sets S, E, B for the solutions of the corresponding operator L to be all algebraic over K? A more precise question is the following version of Dwork's accessory parameter problem: *let V be the set of all operators of order 2 on the curve C, with fixed S and E. Let V_1 be the subset of V corresponding to equations with a full set of algebraic solutions. Does V_1 correspond to an algebraic subset of V?*

Remark 2.1. — In this paper, we shall present a global approach to this type of question. Nevertheless, the following connection with the p-adic operators is worth mentioning. Suppose, for simplicity, that $C = \mathbb{P}^1$ and the coefficients of L are in $\overline{\mathbb{Q}}(x)$. One can reduce the coefficients of L modulo almost all primes of the field of definition of L. Also, one can ask about the p-adic behaviour of the solutions near singular points, for various primes p. If a solution of L is algebraic, then for almost all primes the series representing this solution converges and is bounded by unity in the open p-adic disk $D(0, 1^-)$ of radius unity and centre at the origin (where p is the residue characteristic). Dwork formulated the following conjecture in [**Dwo90**]:

Let V be the set of all operators of order n with coefficients in $\overline{\mathbb{Q}}(x)$, with fixed S and E. Let V_1 the subset of V corresponding to equations where solutions converge in $D(t, 1^-)$ for almost all p. Then V_1 corresponds to an algebraic subset of V.

Here, t is a generic point in some transcendental extension of $\mathbb{Q}_p$, $|t|_p = 1$, such that the residue class of t is transcendental over $\mathbb{F}_p$. On the other hand, if an operator L has a full set of algebraic solutions, then for almost all primes the reduced operator has a full set of solutions or, equivalently, its p-curvature is zero. The celebrated p-curvature conjecture of Grothendieck states that the converse is also true: an operator L has a full set of algebraic solutions if and only if the p-curvature of the reduced operator is zero for almost all primes. For more details on p-adic differential operators, see Dwork [**Dwo81**], [**Dwo90**]. For Katz's proof of Grothendieck's conjecture for Picard-Fuchs operators, see Katz [**Kat72**]. We should also mention (see Honda [**Hon81**] and also Katz [**Kat70**]) that nilpotent p-curvature for almost all p implies that the singularities of a linear operator L are regular. Moreover, if this happens for a set of primes of density 1, then the exponents are rational numbers.

If L is a second order differential operator on C, the following properties are equivalent:

1. - L has a full set of algebraic solutions
2. - the monodromy group of L is finite
3. - the projective monodromy group of L is finite and the Wronskian is an algebraic function over K

In this case, the projective monodromy group is conjugated with the Galois group of the extension $K \subset K(\tau)$, where τ is the ratio of two functions in a base of the space of solutions of L.

The problem of determining the linear operators on $\mathbb{P}^1$ with a full set of algebraic solutions, known in the last decades of the XIX-th century as Fuchs' problem, was solved by Schwarz [**Sch72**] for the hypergeometric operators. Those can be written in the following normalised form:

$$H_{\lambda,\mu,\nu} = D^2 + \frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(x-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4x(x-1)} \tag{2.8}$$

where $\lambda+\mu+\nu>1$. Such an operator has three singular points, 0, 1 and ∞, where the exponent differences $\Delta_{P,H_{\lambda,\mu,\nu}}$ are equal to λ,μ,ν respectively. Using geometric methods and ideas originated in works of Abel and Riemann, Schwarz obtained a table of 15 cases (up to an ordering of λ,μ,ν) when the algebraicity of the solutions is satisfied. He so determined all the second order operators on the projective line, with three singular points and a full set of algebraic solutions.

Schwarz's solution was developed by Klein [**Kle77**], who reduced the list to five essential cases which emphasise the role played by the regular solids. The values of the parameters λ,μ,ν corresponding to hypergeometric operators algebraically integrable, as well as the corresponding projective monodromy groups, are contained in the following table ("the basic Schwarz list"):

(λ,μ,ν)	$\mathbf{G}_{H_{\lambda,\mu,\nu}}$
$(1/n,1,1/n)$	C_n, cyclic of order n
$(1/2,1/n,1/2)$	D_n, dihedral of order $2n$
$(1/2,1/3,1/3)$	$\mathcal{A}_4$, tetrahedral
$(1/2,1/3,1/4)$	$\mathcal{S}_4$, octahedral
$(1/2,1/3,1/5)$	$\mathcal{A}_5$, icosahedral

Klein also proved that the second order linear differential operators with a full set of algebraic solutions are weak pull-backs, by a rational function, of the hypergeometric operators in the basic Schwarz.

At about the same time, Jordan [**Jor78**] noticed that the algebraicity of all the solutions is equivalent to the finiteness of the monodromy group. He approached Fuchs' problem for second and higher order operators by purely group-theoretic means and he proved that the finite subgroups of $GL(n,\mathbb{C})$ could by classified into a finite number of families, similarly to the case $n=2$, when there are two infinite families and three other groups (Jordan's finiteness theorem). For a historic survey of Fuchs' problem, the reader may consult Gray [**Gra86**].

It is not due to the lack of interest in the subject that the case of hypergeometric operators remains, up to our days, the only one where the operators with a full set of algebraic solutions are completely determined. A glance to the formula 1.5 tells us that if L is a second order operator on the projective line with three singular points, then there is no accessory parameter. The operator L is rigid, that is, it is completely determined by the singular points and the local exponents, in other words, by the local data. The reader is referred to Katz [**Kat96**] for more details on the rigidity.

If the accessory parameters are present, the problem becomes much more difficult. And this happens as soon as there is a forth singular point. Along with the p-adic machinery and with group theoretic methods, Klein's results have been, in the last