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THE GROUP THEORY BEHIND MODULAR TOWERS

by

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***Abstract.* —** Geometric considerations identify what properties we desire of the canonical sequence of finite groups that are used to define modular towers. For instance, we need the groups to have trivial center for the Hurwitz spaces in the modular tower to be fine moduli spaces. The Frattini series, constructed inductively, provides our sequence: each group is the domain of a canonical epimorphism, which has elementary abelian p-group kernel, having the previous group as its range. Besides satisfying the desired properties, this choice is readily analyzable with modular representation theory.

Each epimorphism between two groups induces (covariantly) a morphism between the corresponding Hurwitz spaces. Factoring the group epimorphism into intermediate irreducible epimorphisms simplifies determining how the Hurwitz-space map ramifies and when connected components have empty preimage. Only intermediate epimorphisms that have central kernel of order p matter for this. The most important such epimorphisms are those through which the universal central p-Frattini cover factors; the elementary abelian p-Schur multiplier classifies these.

This paper, the second of three in this volume on the topic of modular towers, reviews for arithmetic-geometers the relevant group theory, culminating with the current knowledge of the p-Schur multipliers of our sequence of groups.

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***Résumé* (Théorie des groupes pour les tours modulaires). —** Des considérations géométriques permettent d'identifier quelles propriétés nous souhaitons pour la suite canonique de groupes finis qui sont utilisés pour définir les tours modulaires. Par exemple, les groupes doivent être de centre trivial pour que les espaces de Hurwitz constituant la tour modulaire soient des espaces de modules fins. Notre suite est donnée par la série de Frattini, qui est définie inductivement : chaque groupe est le domaine d'un épimorphisme canonique, lequel a comme noyau un p-groupe abélien élémentaire, et le groupe précédent comme image. En plus de satisfaire les propriétés désirées, ce choix s'interprète naturellement en termes de théorie des représentations modulaires.

Chaque épimorphisme entre deux groupes induit (de manière covariante) un morphisme entre les espaces de Hurwitz correspondants. La factorisation de l'épimorphisme de groupes en épimorphismes irréductibles intermédiaires permet de déterminer plus simplement comment l'application entre espaces de Hurwitz se ramifie et quand les composantes connexes ont des images inverses vides. Pour cela, seuls comptent les épimorphismes intermédiaires qui ont un noyau central d'ordre p. Les plus importants de ces épimorphismes sont ceux à travers lesquels le p-revêtement universel de Frattini se factorise ; ils sont classifiés par le p-groupe élémentaire abélien des multiplicateurs de Schur.

Cet article, le deuxième de trois sur les tours modulaires dans ce volume, revient, à l'intention des arithméticiens-géomètres, sur la théorie des groupes nécessaire à cette théorie, pour aboutir à l'état actuel des connaissances sur les p-groupes de multiplicateurs de Schur de notre suite de groupes.

Contents

1. Introduction

This survey broadly divides into two parts. The first part (§2 and §3) recaps Dèbes' presentation [**Dèb**] of the universal p-Frattini cover and of modular towers. In particular, §2 illustrates difficulties arising from the use of Zorn's lemma in the "top-down" construction of the universal p-Frattini cover, while §3 concentrates on the consequences which the properties of the finite groups G_n have on the modular towers they define. The second part constructs the groups G_n and derives their properties from the "bottom-up", using modular representation theory and, especially, the categorical equivalence of Gruenberg and Roggenkamp [**Gru76**, §10.5]. The appendix displays the functors for this categorical equivalence, since it doesn't seem to be well-known.

Despite relatively few explicit citations herein, many of the results surveyed have been comprehensively catalogued (and produced) by Fried in his work on modular towers. His series of papers on the subject are a primary source: [**Fri95**], [**FK97**], [**Fri02**], [**BF02**], [**Fri**], and [**FS**]. I have tried to introduce required results from modular representation theory steadily but gently; for a general reference, I recommend Benson's text [**Ben98a**].

Before proceeding, recall some elementary categorical definitions.

Definition 1.1. — In any category, for any objects X and Y, a morphism $\phi \in \mathrm{Hom}\,(X,Y)$ is **epic** iff, for all objects Z and for all morphisms $\psi_1, \psi_2 \in \mathrm{Hom}\,(Y,Z)$, if $\psi_1 \circ \phi = \psi_2 \circ \phi$ then $\psi_1 = \psi_2$.

This purely categorical definition is synonymous with "surjective" in the categories of abstract groups, profinite groups, and modules.

Definition 1.2. — An object P of a category $\mathcal{C}$ is **projective** iff, for any objects X and Y of $\mathcal{C}$, any morphism $\psi \in \mathrm{Hom}\,(P,Y)$, and any epic morphism $\phi \in \mathrm{Hom}\,(X,Y)$, there exists a morphism $\pi \in \mathrm{Hom}\,(P,X)$ such that $\phi \circ \pi = \psi$, as illustrated in the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\forall\psi} & Y \\ \Big\downarrow{\scriptstyle \exists\pi} & & \| \\ X & \overset{\forall\phi}{\twoheadrightarrow} & Y \end{array}$$

An object F of $\mathcal{C}$ is **Frattini** iff every morphism to F is epic, i.e., for any object X of $\mathcal{C}$ and any morphism $\phi \in \mathrm{Hom}\,(X,F)$, ϕ is epic.

Given an object X of a category $\mathcal{C}$, a cover of X is defined to be an epic morphism in $\mathrm{Hom}\,(Y,X)$ for some object Y. The collection of covers of X comprise the class of objects of a category whose morphisms are as follows — given two covers, $\phi_1 \in \mathrm{Hom}\,(Y,X)$ and $\phi_2 \in \mathrm{Hom}\,(Z,X)$, $\mathrm{Hom}\,(\phi_1,\phi_2)$ is defined to be the set of morphisms ψ in $\mathrm{Hom}\,(Y,Z)$ such that $\phi_2 \circ \psi = \phi_1$. We also sometimes consider subcategories where we restrict the covers under consideration, but in these cases the set of morphisms between two objects remains the same as in the full category of covers, i.e., these subcategories are full in the technical sense. In the categories of covers we will consider, epic morphisms will always turn out to be surjective. Hence, equivalences between these categories pass along surjectivity of morphisms.

Conventions. The number p is always a positive prime rational integer, G is always a finite group, and k is always a field with characteristic p. The cyclic group of order

n is C_n, the dihedral group of order $2n$ is D_n, the alternating group on n letters is A_n, and the symmetric group on n letters is S_n. The conjugate gag^{-1} of one element a of G by another element g is denoted by ${}^g a$. The commutator $[g,h]$ of two elements g and h of G is $g^{-1}h^{-1}gh$. All modules are finitely generated left-modules. The ring of p-adic integers is denoted by $\mathbb{Z}_p$, and the field with q elements by $\mathbb{F}_q$.

2. The universal p-Frattini cover

Fix a finite group G and consider the category of covers of G within the category of profinite groups; call this category of covers $\mathcal{C}(G)$. A projective Frattini object in this category is called the **universal Frattini cover** of G, as is its domain, which is given the notation $\tilde{G}$. The first construction of this, due to Cossey, Kegel, and Kovács [**CKK80**, Statement 2.4], used Zorn's lemma: projective profinite groups are precisely those isomorphic to closed subgroups of free profinite groups [**FJ05**, Proposition 22.4.7], so take a minimal closed subgroup mapping onto G in any epimorphism onto G with domain a free profinite group. The kernel of the universal Frattini cover is (pro-)nilpotent by the Frattini Argument from which its name derives. Hence, it is the product of its p-Sylows; being closed subgroups of a projective profinite group, they will have to be projective as well, and projective pro-p groups must be free as pro-p groups [**FJ05**, Proposition 22.7.6].

Now consider ${}_p\tilde{G}$, the quotient of $\tilde{G}$ by the p'-Hall subgroup of the kernel of $\tilde{G} \twoheadrightarrow G$, i.e., the product of all of the s-Sylows of the kernel, where s denotes a rational prime distinct from p. This quotient profinite group is called the **universal p-Frattini cover** of G, as is the natural map to G which it inherits. This map is also characterized by being a projective Frattini object in the full subcategory $\mathcal{C}_{p^\infty}(G)$ of $\mathcal{C}(G)$ whose objects are precisely those objects of $\mathcal{C}(G)$ with kernel a pro-p group. The kernel of the universal p-Frattini cover is a free pro-p group called $\mathbf{ker_0}$.

The easiest example is when G is a p-group; then, ${}_p\tilde{G}$ is a free pro-p group with the same minimal number of (topological) generators as G. As a consequence of Schur-Zassenhaus, if G merely has a normal p-Sylow P, then G is a semi-direct product $P \rtimes H$, where $H \simeq G/P$; we say G is **p-split**. When G is p-split, ${}_p\tilde{G} \simeq \hat{F}_n(p) \rtimes H$, where n is the minimal number of generators of the p-Sylow P of G and $\hat{F}_n(p)$ is the pro-p completion of the free group on n generators. The rank (minimal number of topological generators) of $\ker_0$ is $1 + (n-1)|P|$, by the Schreier formula.

Example 2.1. — The alternating group on four elements is isomorphic to $V_4 \rtimes C_3$, where a given generator g of C_3 acts on the Klein four-group V_4 by cyclically permuting the three non-trivial elements. Two (topological) generators a and b of $\hat{F}_2(2)$ may be chosen so that conjugation by g on $\hat{F}_2(2)$ (in ${}_2\tilde{A}_4 \simeq \hat{F}_2(2) \rtimes C_3$) is given by ${}^g a = b$ and ${}^g b = b^{-1}a^{-1}$. Clearly, a and b generate a discrete, dense, free subgroup F_2 of

$\hat{F}_2(2)$ which is stabilized by C_3. We get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_2 & \longrightarrow & F_2 \rtimes C_3 & \longrightarrow & C_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \hat{F}_2(2) & \longrightarrow & {}_2\tilde{A}_4 & \longrightarrow & C_3 & \longrightarrow & 1 \end{array}$$

By the Schreier formula, $\ker_0$ has rank 5 and its intersection with F_2 is a free group F_5 of rank 5, normal inside of F_2. There is another commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_5 & \longrightarrow & F_2 \rtimes C_3 & \longrightarrow & A_4 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_4 & \longrightarrow & A_4 & \longrightarrow & 1 \end{array}$$

where the vertical maps are dense group monomorphisms.

In general, the approach we've been following so far fails to provide detailed information about the universal p-Frattini cover, the preceding example being a rare counterexample describable by a discrete analogue. Even p-split groups can often not be described this way. One reason to expect this failure is the non-constructiveness of using Zorn's lemma to create the universal cover. Consider two examples illustrating the limitations.

Example 2.2. — Our first example comes from Holt and Plesken [**HP89**]. Embedding A_4 into A_5 leads to an embedding of ${}_2\tilde{A}_4$ into ${}_2\tilde{A}_5$ and the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_4 & \longrightarrow & A_4 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_5 & \longrightarrow & A_5 & \longrightarrow & 1 \end{array}$$

The leftmost vertical map is an isomorphism. However, there is NO group Γ which can fit into a commutative diagram of exact sequences of the following form, where the vertical maps are dense monomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_5 & \longrightarrow & \Gamma & \longrightarrow & A_5 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_5 & \longrightarrow & A_5 & \longrightarrow & 1 \end{array}$$

The proof examines the character of the 2-adic Frattini lattice (cf. §7) of $\mathrm{SL}_2(\mathbb{F}_5)$ and is beyond the scope of these limited notes.

Example 2.3. — A result of Dyer and Scott [**DS75**] says that, for any automorphism σ of prime order s acting on a discrete free group F, there is a basis X of F such that one of the following holds for every x in X:

i) $\sigma(x) = x$