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FAMILIES OF LINEAR DIFFERENTIAL EQUATIONS ON THE PROJECTIVE LINE

by

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Abstract. — The aim is to extend results of M.F. Singer on the variation of differential Galois groups. Let C be an algebraically closed field of characteristic 0. One considers certain families of connections of rank n on the projective line parametrized by schemes X over C. Let $G \subset \mathrm{GL}_n$ be an algebraic subgroup. It is shown that $X(=G)$, the set of closed points with differential Galois group G, is constructible for all families if and only if G satisfies a condition introduced by M.F. Singer. For the proof, techniques for handling families of vector bundles and connections are developed.

Résumé **(Familles d'équations différentielles linéaires sur la droite projective)**
Le but est de compléter des résultats de M.F. Singer concernant la variation des groupes de Galois différentiels. Soit C un corps algébriquement clos, de caractéristique 0. On considère des familles de connections de rang n sur la droite projective, paramétrisées par des schémas X sur C. Soit $G \subset \mathrm{GL}_n$ un sous-groupe algébrique. On montre que $X(=G)$, l'ensemble des points fermés de X avec G comme groupe de Galois différentiel, est constructible pour toute famille si et seulement si le groupe G satisfait une condition introduite par M.F. Singer. Pour la démonstration, des techniques concernant des familles de fibrés vectoriels et des connections sont développées.

1. Introduction

C is an algebraically closed field of characteristic 0 and X denotes a scheme of finite type over C. We fix a vector space V of dimension n over C and an algebraic subgroup G of $\mathrm{GL}(V)$. We will define *families of linear differential equations* on the projective line C, parametrized by X. These families are of a more general nature than the moduli spaces, defined in [**Ber02**]. For each closed point x of X (i.e., $x \in X(C)$), the differential equation corresponding to x has a differential Galois group, denoted by $\mathrm{Gal}(x)$. It is shown that the condition "$\mathrm{Gal}(x) \subset G$" for closed points x of X defines a closed subset of X. This generalizes Theorem 4.2 of [**Ber02**], where this statement is proved for moduli spaces.

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The aim is to show that the set of closed points $x \in X$ for which the differential Galois group $\mathrm{Gal}(x)$ of the corresponding equation is equal to G is a *constructible* subset of X, i.e., of the form $\cup_{i=1}^{n}(O_i \cap F_i)$ for open sets O_i and closed sets F_i. This statement (and the earlier one) has to be made more precise by providing a suitable definition of "family of differential equations" and a meaning for the expression $\mathrm{Gal}(x) \subset G$. Moreover, a condition on the group G is essential.

In his paper [**Sin93**], M.F. Singer defines a set of differential operators, by giving some local data. He proves that under a certain condition on G, the subset of the differential equations with Galois group equal to (a conjugate of) G is constructible. This condition on G will be called the *Singer condition*. We consider the same problem, in our context of families of differential equations parametrized by a scheme X. We will construct for any group G that does not satisfy the "Singer condition" an example of a moduli family $\mathbb{M}$ such that $\{x \in \mathbb{M} \mid \mathrm{Gal}(x) = G\}$ is not constructible. Finally, from these constructions one deduces an alternative description of the Singer condition.

2. The Singer condition

Let G be a linear algebraic group over C. First we will recall the Singer condition on G, as given in [**Sin93**]. A character χ of G is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$, where $\mathbb{G}_m$ stands for the multiplicative group C^*. The set $X(G)$ of all characters is a finitely generated abelian group. Let $\ker X(G)$ denote the intersection of the kernels of all $\chi \in X(G)$. This intersection is a characteristic (closed) subgroup of G. As usual, G^o denotes the connected component of the identity of G. The group $\ker X(G^o)$ is a normal, closed subgroup of G^o and of G. Let $\chi_1, \ldots, \chi_s$ generate $X(G^o)$. Then $\ker X(G^o)$ is equal to the intersection of the kernels of $\chi_1, \ldots, \chi_s$. In other words $\ker X(G^o)$ is the kernel of the morphism $G^o \to \mathbb{G}_m^s$, given by $g \mapsto (\chi_1(g), \ldots, \chi_s(g))$. The image is a connected subgroup of $\mathbb{G}_m^s$ and therefore a torus T. Hence $G^o/\ker X(G^o)$ is isomorphic to T. Moreover, by definition, T is the largest torus factor group of G^o. One considers the exact sequence:

$$1 \longrightarrow G^0/\ker X(G^0) \longrightarrow G/\ker X(G^0) \longrightarrow G/G^0 \longrightarrow 1.$$

Since $G^0/\ker\ X(G^0)$ is abelian, this sequence induces an action of G/G^0 on $G^0/\ker X(G^0)$ by conjugation.

Definition 2.1. — A linear algebraic group G satisfies the *Singer Condition* if the action of G/G^0 on $G^0/\ker X(G^0)$ is trivial.

The Singer condition can be stated somewhat simpler, using $U(G) \subset G$, the subgroup generated by all unipotent elements in G.

Lemma 2.2. — $U(G) = U(G^o)$ *is equal to* $\ker X(G^o)$ *and the Singer condition is equivalent to "*$G^o/U(G)$ *lies in the center of* $G/U(G)$*".*

Proof. — Fix an embedding $G \subset \mathrm{GL}(V)$, where V is a finite dimensional vector space over C. First we prove that $U(G)$ is a closed connected normal subgroup of G. Let $I+B,\ B \neq 0$ be a unipotent element of G. Then $I+B = e^D$, for some nilpotent element $D = \sum(-1)^{i-1}\frac{B^i}{i} \in \mathrm{End}(V)$. The Zariski closure $\overline{\{(I+B)^n | n \in \mathbb{Z}\}}$ of the group generated by $I+B$ lies in G and is equal to the group $\{e^{tD} | t \in C\}$, which is isomorphic to the additive group $\mathbb{G}_a$ over C. Hence $U(G)$ is generated by these connected subgroups of G and by Proposition 2.2.6 of [**Spr98**] the group $U(G)$ is closed and connected. Further $U(G)$ is a normal subgroup and even a characteristic subgroup, since the set of unipotent elements of G is stable under any automorphism of G. The connectedness of $U(G)$ implies $G^o \supset U(G) = U(G^o)$.

Now we will show that $G^o/U(G^o)$ is a torus. Since the *unipotent radical* $R_u(G^o)$ lies in $U(G^o)$, we may divide G^o by $R_u(G^o)$ and assume that G^o to be *reductive*. Then by [**Spr98**][corollary 8.1.6] we have $G^o = R(G^o)\cdot(G^o, G^o)$, where $R(G^o)$ is the radical of G^o, and where (G^o, G^o) is the *commutator subgroup* of G^o. The latter group is a semi-simple subgroup, according to the same corollary. By [**Spr98**][theorem 8.1.5] we get that (G^o, G^o) is generated by unipotent elements, so $(G^o, G^o) \subset U(G^o)$. Since $R(G^o)$ is a torus, its image $G^o/U(G^o)$ is a torus, too. This proves $U(G^o) \supset \ker X(G^o)$. The other inclusion follows from the observation that every unipotent element lies in the kernel of every character.

Finally, the triviality of the action of G/G^o on $G^o/U(G^o)$ is clearly equivalent to $G^o/U(G^o)$ lies in the center of $G/U(G^o)$. □

Remarks 2.3

(1) Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup. *For the moment we admit the following items* (see 3.4, 3.5 (2), 4.1 and 4.2):

- The definition of a family of differential equations, parametrized by X.
- The meaning of $\mathrm{Gal}(x) \subset G$ for $x \in X(C)$.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset G\}$ is closed.
- $\{x \in X(C) \mid \mathrm{Gal}(x) \subset hGh^{-1}$ for some $h \in \mathrm{GL}(V)\}$ is constructible.

Consider the following finiteness condition for the group G: $(*)$ G has finitely many proper closed subgroups $H_1, \ldots, H_s$, such that every proper closed subgroup is contained in a conjugate of one of the H_i. One easily deduces: *If G satisfies $(*)$, then $\{x \in X(C) \mid \mathrm{Gal}(x) = G\}$ is constructible.*

(2) *If G satisfies $(*)$, then $G/U(G)$ is a finite group and in particular G satisfies the Singer condition.* Indeed, $(*)$ also holds for $G/U(G)$. If $T := G^o/U(G) \neq \{1\}$, then one can produce infinitely many proper normal subgroups of $G/U(G)$. Namely, for any integer $m > 1$ the subgroup $T[m]$, consisting of the m-torsion elements of T, is a normal subgroup. One concludes that $G/U(G)$ is finite.

(3) Consider $G := \mathrm{SL}_2(C)$. The classification of the proper closed subgroups H of G states that H is either contained in a Borel subgroup or in a conjugate of the

infinite dihedral group $D_\infty^{\mathrm{SL}_2}$ or is conjugated to one of the special finite groups: the tetrahedral group, the octahedral group, the icosahedral group. Thus G satisfies $(*)$ and moreover, $G/U(G) = \{1\}$.

(4) The infinite dihedral group $G = D_\infty^{\mathrm{SL}_2}$ has the properties: $G^o = \mathbb{G}_m$, $U(G^o) = 1$ and G/G^o acts non-trivially on G^o. Thus G does not satisfy the Singer condition. For this group one can produce moduli spaces $\mathbb{M}$ such that $\{x \in \mathbb{M}(C) \mid \mathrm{Gal}(x) = G\}$ is not constructible (see example 2.6).

(5) For the following two examples, namely moduli spaces and the groups $\mathbb{G}_a^3$ and, $\mathbb{G}_m^n$, the Singer condition is valid, but $(*)$ does not hold. We will show explicitly that these groups define constructible subsets.

***Example 2.4* (A moduli space with differential Galois groups in $\mathbb{G}_a^3$)**

Moduli spaces of the type considered here are defined in [**Ber02**]. V is a 4-dimensional vector space over C with basis $e_1, \ldots, e_4$. The element $N \in \mathrm{End}(V)$ is given by $N(e_i) = 0$ for $i = 1, 2, 3$ and $N(e_4) = e_1$. The data for the moduli problem are:

- Three distinct singular points $a_1, a_2, a_3 \in C^*$. The point ∞ is allowed to have a, non prescribed, regular singularity.
- For each singular point a_i, the differential operator $\frac{d}{d(z-a_i)} + \frac{N}{z-a_i}$.

Some calculations lead to an identification $\mathrm{GL}(4, C) \times \mathrm{GL}(4, C) \to \mathbb{M}$, where $\mathbb{M}$ is the moduli space of the problem. Let $m := (\phi_2, \phi_3)$ denote a closed point of the first space, then the corresponding universal differential operator is

$$\frac{d}{dz} + \frac{N}{z - a_1} + \frac{\phi_2 N \phi_2^{-1}}{z - a_2} + \frac{\phi_3 N \phi_3^{-1}}{z - a_3}.$$

Let $G := \mathbb{G}_a^3$ the subgroup of $\mathrm{GL}(V)$ consisting of the maps of the form $I + B$, $Be_1 = 0$ and $Be_i \in Ce_1$ for $i = 2, 3, 4$. The condition $\mathrm{Gal}(m) \subset \mathbb{G}_a^3$ can be seen to be equivalent to $\phi_2(e_1), \phi_3(e_1) \in Ce_1$. This describes the set $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ completely. The above differential operator evaluated at a point of $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ has the form

$$\frac{d}{dz} + \begin{pmatrix} 0 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ where}$$

$$(h_1, h_2, h_3) = \frac{1}{z - a_1}(0, 0, 1) + \frac{1}{z - a_2}(f_1, f_2, f_3) + \frac{1}{z - a_3}(g_1, g_2, g_3).$$

Moreover, f_1, f_2, f_3 are polynomials of degree ≤ 2 in the entries of ϕ_2 and the g_1, g_2, g_3 are polynomials of degree ≤ 2 in the entries of ϕ_3.

Now G has infinitely many (non-conjugated) maximal proper closed subgroups and there is no obvious reason why $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = G\}$ should be constructible. We continue the calculation. The differential Galois group $\mathrm{Gal}(m)$, with m such that

$\mathrm{Gal}(m) \subset G$, is in fact the differential Galois group for the three inhomogeneous equations $y_i' = h_i$, $i = 1, 2, 3$ over $C(z)$. Thus $\mathrm{Gal}(m)$ is a proper subgroup of G if and only if there is a non trivial linear combination $c_1h_1 + c_2h_2 + c_3h_3$ with $c_1, c_2, c_3 \in C$ such that $y' = c_1h_1 + c_2h_2 + c_3h_3$ has a solution in $C(z)$. Now y exists if and only if $c_1h_1 + c_2h_2 + c_3h_3$ has residue 0 at the points a_1, a_2, a_3. The existence of such a linear combination translates into a linear dependence and the explicit equation $f_1(a_2)g_2(a_3) - f_2(a_2)g_1(a_3) = 0$. This defines a closed subset of $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ and so $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = G\}$ is constructible. We note that every linear subspace of $\mathbb{G}_a^3 \cong C^3$, which contains $(0, 0, 1)$, occurs as differential Galois group.

Example 2.5 **(A moduli space with differential Galois groups in $\mathbb{G}_m^n$)**
The data for the moduli problem are:

- A vector space V of dimension n over C and basis $e_1, \ldots, e_n$.
- Singular points $a_1, \ldots, a_s$, different from 0 and ∞, We allow ∞ to have a non-prescribed regular singularity.
- Local differential operators $\frac{d}{d(z-a_i)} + \frac{M_i}{z-a_i}$, where $e_1, \ldots, e_n$ are eigenvectors for all $M_i \in \mathrm{End}(V)$.

The moduli space $\mathbb{M}$ can be identified with $\mathrm{GL}(V)^{s-1}$. At a closed point $m = (\phi_2, \ldots, \phi_s) \in \mathrm{GL}(V)^{s-1}$ the universal differential operator reads

$$\frac{d}{dz} + \sum_{i=1}^{s} \frac{\phi_i M_i \phi_i^{-1}}{z - a_i},$$

where $\phi_1 = I$. The group $\mathbb{G}_m^n \cong G \subset \mathrm{GL}(V)$ consists of the maps for which each e_i is an eigenvector. Above the closed subset $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ the differential operator has the form

$$L := \frac{d}{dz} + \sum_{i=1}^{s} \frac{N_i}{z - a_i},$$

with $N_1 = M_1$ and each N_i is a diagonal matrix w.r.t. the basis $e_1, \ldots, e_n$ and having the same eigenvalues as M_i. The space $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) \subset G\}$ has positive dimension and is rather large if there is at least one M_i with $i > 1$ having an eigenvalue with multiplicity > 1. However the number of differential operators L is finite! Thus only a finite number of algebraic subgroups of $G \cong \mathbb{G}_m^n$ occur as differential Galois group $\mathrm{Gal}(m)$. One concludes that for every algebraic subgroup $H \subset G$, the set $\{m \in \mathbb{M} \mid \mathrm{Gal}(m) = H\}$ is constructible.

This example is the general pattern for "families" with differential Galois groups contained in some torus T. Again, there are only finitely many distinct differential operators L possible and therefore only finitely many possibilities for the differential Galois group. This implies that for every algebraic subgroup $H \subset T$ the set of the points with differential Galois group equal to H is constructible.