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BRIEF INTRODUCTION TO PAINLEVÉ VI

by

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Abstract. — We will give a quick introduction to isomonodromy and the sixth Painlevé differential equation, leading to some questions regarding algebraic solutions.

***Résumé* (Une brève introduction à Painlevé VI).** — Nous donnons une brève introduction à l'isomonodromie et à la sixième équation différentielle de Painlevé, ce qui conduit à des questions sur les solutions algébriques.

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1. Introduction

The sixth Painlevé equation ($\mathrm{P_{VI}}$) is a second order nonlinear differential equation for a complex function $y(t)$:

$$y'' = R(y, y', t)$$

where R is a certain rational function (see below) depending on four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. (Thus we need to fix these parameters to get a particular $\mathrm{P_{VI}}$ equation.) The main thing one needs to know about $\mathrm{P_{VI}}$ is the following:

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Fact. — Suppose we have a local solution y of $\mathrm{P_{VI}}$ on some disk $D \subset \mathbb{P}^1 \setminus \{0,1,\infty\}$ in the three-punctured sphere. Then y extends, as a solution of $\mathrm{P_{VI}}$, to a meromorphic function on the universal cover of $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

Thus solutions only branch at $0,1,\infty$ and all other singularities are just poles; this is the so-called 'Painlevé property' of the equation.

Thus $\mathrm{P_{VI}}$ shares many of the properties of the Gauss hypergeometric equation, which is a *linear* second order equation whose solutions branch only at $0,1,\infty$.

Another well-known fact about $\mathrm{P_{VI}}$ is that generic solutions $y(t)$ of $\mathrm{P_{VI}}$ are "new" transcendental functions (i.e., they are not expressible in terms of classical special functions). Thus it is very difficult to find explicit solutions to $\mathrm{P_{VI}}$ in general.

However, for special values of the parameters it turns out that there are explicit solutions, and even solutions $y(t)$ which are algebraic, i.e., defined implicitly by polynomial equations

$$F(y,t)=0. \tag{1}$$

Our aim is to describe some of the geometry behind $\mathrm{P_{VI}}$ leading up to a description of how some of these algebraic solutions may be constructed.

Note immediately that by definition such plane algebraic curves

$$\{(y,t) \mid F(y,t)=0\} \subset \mathbb{C}^2$$

are covers of the t-line, branched only at $0,1,\infty$ and so are Belyi curves. Also, in all examples so far, the polynomial F turns out to have integer coefficients.

To give a brief taste of the geometry let us mention that, as is often the case, the three-punctured sphere above arises as the moduli space of (ordered) four-tuples of points on another $\mathbb{P}^1$. Explicitly, to each $t \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ we will associate the four-tuple $(0,t,1,\infty)$ of points and in turn the four-punctured sphere

$$\mathbb{P}_t := \mathbb{P}^1 \setminus \{0,t,1,\infty\}.$$

As we will explain, $\mathrm{P_{VI}}$ arises by considering (isomonodromic) deformations of certain non-rigid linear differential equations on theses four-punctured spheres. In particular solving $\mathrm{P_{VI}}$ leads to explicit linear differential equations on the four-punctured sphere with known, non-rigid, monodromy representations.

Acknowledgments. — The reader should note that the literature on $\mathrm{P_{VI}}$ is huge and we will not attempt a survey. (A good bibliography and historical survey may be found in [**DM00**].) This note is written to explain some introductory facts about the method of [**Boa05**], which extends that of Dubrovin and Mazzocco [**DM00**]. I would like to thank Daniel Bertrand and Pierre Dèbes for the invitation to speak at this conference.

2. Monodromy and actions of the fundamental group of the base

Suppose we have a complete flat connection on a fibre bundle $\pi : M \to B$. Choose a basepoint $t \in B$ and let $M_t = \pi^{-1}(t)$ be the fibre of M over t. (See appendix B.)

Then given any loop γ in B based at t, we may integrate the connection on M around γ, yielding an automorphism

$$a_\gamma : M_t \xrightarrow{\cong} M_t$$

of the fibre over t. This automorphism only depends on the homotopy class of the loop γ (since the connection is flat), and in this way one obtains an action of the fundamental group of the base on the fibre, i.e., a homomorphism

$$\pi_1(B) \longrightarrow \mathrm{Aut}(M_t),$$

the *monodromy action*.

This should be compared with the cases of a) linear connections (where the fibre is a vector space V and so one obtains a *representation* $\pi_1(B) \to \mathrm{GL}(V)$), and b) coverings (where the fibre is a finite set and so $\mathrm{Aut}(M_t) = \mathrm{Sym}_n$).

We will be interested in horizontal sections of such flat connections which are finite covers of the base — i.e., sections which only have a finite number of branches. The point to be made here is that, in terms of the monodromy action, such sections correspond precisely to the *finite orbits* of the monodromy action. Given a point of $m \in M_t$ which is in a finite orbit, the horizontal section of the connection through m will extend, by definition, to a section with a finite number of branches.

3. Main example: the $\mathbf{P_{VI}}$ fibrations

The main example of fibre bundle with complete flat connection we are interested in here comes from geometry. It is the simplest *isomonodromy* or *non-abelian Gauss-Manin* connection.

Take the base B to be the three-punctured sphere

$$B := \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

For each point $t \in B$ there is a corresponding four-punctured sphere, namely

$$\mathbb{P}_t := \mathbb{P}^1 \setminus \{0, t, 1, \infty\}.$$

Thus we can think of B as parameterising a (universal) family of four-punctured spheres, with labelled punctures. Write a_1, a_2, a_3, a_4 for these punctures positions:

$$(a_1, a_2, a_3, a_4) := (0, t, 1, \infty).$$

For each $t \in B$ we consider the space of conjugacy classes of $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group of $\mathbb{P}_t$

$$\mathrm{Hom}(\pi_1(\mathbb{P}_t), G)/G \tag{2}$$

where $G := \mathrm{SL}_2(\mathbb{C})$, and we have not specified the basepoint used in $\pi_1(\mathbb{P}_t)$, since changing basepoints yields conjugate representations (which are identified in the quotient (2)).

Now suppose we choose four generic conjugacy classes of $G = \mathrm{SL}_2(\mathbb{C})$

$$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \subset G.$$

Then we can consider the subset of (2),

$$\boldsymbol{\mathcal{C}}_t := \mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{P}_t), G)/G \subset \mathrm{Hom}(\pi_1(\mathbb{P}_t), G)/G$$

of representations which take simple positive loops around a_i into $\mathcal{C}_i$ for $i = 1, 2, 3, 4$.

Explicitly if we choose loops γ_i generating $\pi_1(\mathbb{P}_t)$ such that $\gamma_4 \cdot \gamma_3 \cdot \gamma_2 \cdot \gamma_1$ is contractible and that γ_i is a simple positive loop around a_i. Then each $\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}_t), G)$ determines matrices $M_i = \rho(\gamma_i) \in G$ and we obtain the explicit description:

$$\boldsymbol{\mathcal{C}}_t \cong \{(M_1, M_2, M_3, M_4) \bigm| M_i \in \mathcal{C}_i, M_4 \cdots M_1 = 1\}/G \tag{3}$$

where G acts by overall conjugation. A simple dimension count shows that in general these spaces are of complex dimension two and taking the invariant functions identifies $\boldsymbol{\mathcal{C}}_t$ with an affine cubic surface, (cut out by the so-called "Fricke relation" between the invariants) which is smooth in general (see e.g. [**Iwa02, Boa05**]).

Remark. — One might ask why, in the simplest case, one cannot have dimension one instead, but that is because these spaces of "conjugacy classes of fundamental group representations with fixed local conjugacy classes", have natural holomorphic symplectic structures on them, so are even-dimensional.

Lemma. — *The surfaces $\boldsymbol{\mathcal{C}}_t$ fit together as the fibres of a (nonlinear) fibre bundle*

$$M \longrightarrow B$$

over B and this fibration has a natural complete flat connection defined by identifying representation with the "same" monodromy.

Proof. — Choose $t \in B$ arbitrarily and choose loops generating $\pi_1(\mathbb{P}_t)$ to obtain an explicit description of $\boldsymbol{\mathcal{C}}_t$ as in (3). Then there is a small neighbourhood U of t in B for which we can use the *same loops* to generate $\pi_1(\mathbb{P}_s)$ for any $s \in U$. Thus we have isomorphisms between $\boldsymbol{\mathcal{C}}_s$ and the right-hand side of (3) for any $s \in U$. This gives a preferred trivialisation of M over U (and one obtains the same trivialisation if different loops were initially chosen). Since t was arbitrary we may cover B with such patches U with a preferred trivialisation over each. This is equivalent to giving a complete flat connection. □

Thus we are now in the situation of the previous section, with a complete flat connection on a fibre bundle.

The Painlevé VI equation amounts to an explicit description of this connection. Very briefly one defines two specific functions y, x on a dense open subset of M, which restrict to local coordinates on each fibre. (See appendix A for a better approximation.) Writing out the connection in these coordinates yields a pair of coupled first order non-linear differential equations for $y(t), x(t)$. Eliminating x then yields a second order equation, the $\mathrm{P_{VI}}$ equation, for $y(t)$:

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$

Thus the time t in $\mathrm{P_{VI}}$ is essentially the cross-ratio of the four pole positions (and is the coordinate t on the three-punctured sphere B). Also the four parameters $\alpha, \beta, \gamma, \delta$ in $\mathrm{P_{VI}}$ correspond to the choice of four conjugacy classes $\mathcal{C}_i \subset \mathrm{SL}_2(\mathbb{C})$.

The main point is that from this geometrical viewpoint we see that that branching of solutions $y(t)$ to $\mathrm{P_{VI}}$ corresponds to the monodromy of the connection on $M \to B$. Since this connection is complete, its monodromy amounts to an action of the fundamental group of B on a fibre $\boldsymbol{\mathcal{C}}_t$.

In particular finite-branching solutions of $\mathrm{P_{VI}}$ will be defined on finite covers of B (i.e covers of $\mathbb{P}^1$ branched only over $0, 1, \infty$) and will correspond to finite orbits of the monodromy action.

Explicitly this monodromy action can be described as follows in terms of the standard Hurwitz action.

The three-string braid group B_3 acts on $G^3 = G \times G \times G$ as follows

$$\begin{aligned} \beta_1(M_3, M_2, M_1) &= (M_2, M_2^{-1}M_3M_2, M_1) \\ \beta_2(M_3, M_2, M_1) &= (M_3, M_1, M_1^{-1}M_2M_1) \end{aligned} \tag{4}$$

where $M_i \in G$. The fundamental group of the base B is the free group on two letters $\pi_1(B) = F_2$ and this appears as the subgroup $< \beta_1^2, \beta_2^2 >$ of B_3. This F_2 action on G^3 restricts and descends to an action on $\boldsymbol{\mathcal{C}}_t$ (where the M_i arise as in (3)). Explicitly, with our conventions, the generator β_1^2 corresponds to the monodromy of y around 1 and β_2^2 to the monodromy of y around 0. An equivalent way of thinking of this is to observe this F_2 also arises as the pure mapping class group of the four-punctured sphere, which acts on the conjugacy classes of representations in the natural way, by pullback [**Boa06**].

4. Algebraic solutions

The problem of finding algebraic solutions to $\mathrm{P_{VI}}$ can be broken into two parts:

1) Find all the finite orbits of the explicit braid group action (4) on triples of elements of $\mathrm{SL}_2(\mathbb{C})$. (Since all algebraic solutions will be finite branching these orbits will a priori contain the branches of all algebraic solutions.)