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SIX RESULTS ON PAINLEVÉ VI

by

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Abstract. — After recalling some of the geometry of the sixth Painlevé equation, we describe how the Okamoto symmetries arise naturally from symmetries of Schlesinger's equations and summarise the classification of the Platonic Painlevé six solutions.

***Résumé* (Six résultats sur Painlevé VI).** — Après quelques rappels sur la géométrie de la sixième équation de Painlevé, nous expliquons comment les symétries d'Okamoto résultent de façon naturelle des symétries des équations de Schlesinger et comment elles conduisent à la classification des solutions platoniques de la sixième équation de Painlevé.

1. Background

The Painlevé VI equation is a second order nonlinear differential equation which governs the isomonodromic deformations of linear systems of Fuchsian differential equations of the form

$$\frac{d}{dz} - \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1}\right), \qquad A_i \in \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C}) \tag{1}$$

as the second pole position t varies in $B := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. (The general case —varying all four pole positions— reduces to this case using automorphisms of $\mathbb{P}^1$.)

By 'isomonodromic deformation' one means that as t varies the linear monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}) \to \mathrm{SL}_2(\mathbb{C})$$

of (1) does not change (up to overall conjugation). Of course, this is not quite well-defined since as t varies one is taking fundamental groups of different four-punctured

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spheres, and it is crucial to understand this in order to understand the global behaviour (nonlinear monodromy) of Painlevé VI solutions. For small changes of t there are canonical isomorphisms between the fundamental groups: if t_1, t_2 are in some disk $\Delta \subset B$ in the three-punctured sphere then one has a canonical isomorphism

$$\pi_1(\mathbb{P}^1 \setminus \{0, t_1, 1, \infty\}) \cong \pi_1(\mathbb{P}^1 \setminus \{0, t_2, 1, \infty\})$$

coming from the homotopy equivalences

$$\mathbb{P}^1 \setminus \{0, t_1, 1, \infty\} \hookrightarrow \{(t, z) \in \Delta \times \mathbb{P}^1 \big| z \neq 0, t, 1, \infty\} \hookleftarrow \mathbb{P}^1 \setminus \{0, t_2, 1, \infty\}.$$

(Here we view the central space as a family of four-punctured spheres parameterised by $t \in \Delta$ and are simply saying that it contracts onto any of its fibres.)

In turn, by taking the space of such ρ's, i.e., the space of conjugacy classes of $\mathrm{SL}_2(\mathbb{C})$ representations of the above fundamental groups, one obtains canonical isomorphisms:

$$\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t_1, 1, \infty\}), G)/G \;\cong\; \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t_2, 1, \infty\}), G)/G$$

where $G = \mathrm{SL}_2(\mathbb{C})$. Geometrically this says that the spaces of representations

$$\widetilde{M}_t := \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}), G)/G$$

constitute a 'local system of varieties' parameterised by $t \in B$. In other words, the natural fibration

$$\widetilde{M} := \{(t, \rho) \mid t \in B, \rho \in \widetilde{M}_t \} \longrightarrow B$$

over B (whose fibre over t is $\widetilde{M}_t$) has a natural flat (Ehresmann) connection on it. Moreover, this connection is complete: over any disk in B any two fibres have a canonical identification.

To get from here to Painlevé VI ($\mathrm{P_{VI}}$) one pulls back the connection on the fibre bundle $\widetilde{M}$ along the Riemann–Hilbert map and writes down the resulting connection in certain coordinates. Consequently we see immediately that the monodromy of $\mathrm{P_{VI}}$ solutions corresponds (under the Riemann–Hilbert map) to the monodromy of the connection on the fibre bundle $\widetilde{M}$. However, since this connection is flat and complete, its monodromy is given by the action of the fundamental group of the base $\pi_1(B) \cong \mathcal{F}_2$ (the free group on 2 generators) on a fibre $\widetilde{M}_t \subset \widetilde{M}$, which can easily be written down explicitly.

Before describing this in more detail let us first restrict to linear representations ρ having local monodromies in fixed conjugacy classes:

$$M_t := \{\rho \in \widetilde{M}_t \mid \rho(\gamma_i) \in \mathcal{C}_i, i = 1, 2, 3, 4\} \subset \widetilde{M}_t$$

where $\mathcal{C}_i \subset G$ are four chosen conjugacy classes, and γ_i is a simple positive loop in $\mathbb{P}^1 \setminus \{0, t, 1, \infty\}$ around a_i, where $(a_1, a_2, a_3, a_4) = (0, t, 1, \infty)$ are the four pole positions. (By convention we assume the loop $\gamma_4 \cdots \gamma_1$ is contractible, and note that M_t is two-dimensional in general.) The connection on $\widetilde{M}$ restricts to a (complete flat Ehresmann) connection on the fibration

$$M := \{(t, \rho) \mid t \in B, \rho \in M_t \} \to B$$

whose fibre over $t \in B$ is M_t. The action of $\mathcal{F}_2 = \pi_1(B)$ on the fibre M_t (giving the monodromy of the connection on the bundle M and thus the monodromy of the corresponding P_{VI} solution) is given explicitly as follows. Let w_1, w_2 denote the generators of $\mathcal{F}_2$, thought of as simple positive loops in B based at $1/2$ encircling 0 (resp. 1) once. Then, w_i acts on $\rho \in M_t$ as the square of ω_i where ω_i acts by fixing M_j for $j \neq i, i+1,\ (1 \leqslant j \leqslant 4)$ and

$$\omega_i(M_i,\ M_{i+1}) = (M_{i+1},\ M_{i+1}M_iM_{i+1}^{-1}) \tag{2}$$

where $M_j = \rho(\gamma_j) \in G$ is the jth monodromy matrix. Indeed, $\mathcal{F}_2$ can naturally be identified with the pure mapping class group of the four-punctured sphere and this action comes from its natural action (by push-forward of loops) as outer automorphisms of $\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\})$, cf. [**5**]. (The geometric origins of this action in the context of isomonodromy can be traced back at least to Malgrange's work [**28**] on the global properties of the Schlesinger equations.)

On the other side of the Riemann–Hilbert correspondence we may choose some adjoint orbits $\mathcal{O}_i \subset \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ such that

$$\exp(2\pi\sqrt{-1}\mathcal{O}_i) = \mathcal{C}_i$$

and construct the space of residues:

$$\boldsymbol{\mathcal{O}} := \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 /\!\!/ G = \left\{\ (A_1, \dots A_4) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 \ \Big|\ \sum A_i = 0\ \right\} / G$$

where, on the right-hand side, G is acting by diagonal conjugation: $g \cdot (A_1, \dots A_4) = (gA_1g^{-1}, \dots, gA_4g^{-1})$. This space $\boldsymbol{\mathcal{O}}$ is also two-dimensional in general. To construct a Fuchsian system (1) out of such a four-tuple of residues one must also choose a value of t, so the total space of linear connections we are interested in is:

$$\mathcal{M}^* := \boldsymbol{\mathcal{O}} \times B$$

and we think of a point $(\mathbf{A}, t) \in \mathcal{M}^*$, where $\mathbf{A} = (A_1, \dots, A_4)$, as representing the linear connection

$$\nabla = d - A dz, \qquad \text{where } A = \sum_1^3 \frac{A_i}{z - a_i}, \quad (a_1, a_2, a_3, a_4) = (0, t, 1, \infty)$$

or equivalently the Fuchsian system (1).

If we think of $\mathcal{M}^*$ as being a (trivial) fibre bundle over B with fibre $\boldsymbol{\mathcal{O}}$ then, provided the residues are sufficiently generic (e.g., if no eigenvalues differ by positive integers), the Riemann–Hilbert map (taking linear connections to their monodromy representations) gives a bundle map

$$\nu : \mathcal{M}^* \to M.$$

Written like this the Riemann–Hilbert map ν is a holomorphic map (which is in fact injective if the eigenvalues are also nonzero cf. e.g., [**25**, Proposition 2.5]). We may then pull-back (restrict) the nonlinear connection on M to give a nonlinear connection on the bundle $\mathcal{M}^*$, which we will refer to as the *isomonodromy connection*.

The remarkable fact is that even though the Riemann–Hilbert map is transcendental, the connection one obtains in this way is algebraic. Indeed Schlesinger [**31**] showed that locally horizontal sections $\mathbf{A}(t) : B \to \mathcal{M}^*$ are given (up to overall conjugation) by solutions to the Schlesinger equations:

$$\frac{dA_1}{dt} = \frac{[A_2, A_1]}{t}, \qquad \frac{dA_2}{dt} = \frac{[A_1, A_2]}{t} + \frac{[A_3, A_2]}{t-1}, \qquad \frac{dA_3}{dt} = \frac{[A_2, A_3]}{t-1} \tag{3}$$

which are (nonlinear) *algebraic* differential equations.

To get from the Schlesinger equations to $\mathrm{P_{VI}}$ one proceeds as follows (cf. [**24**, Appendix C]). Label the eigenvalues of A_i by $\pm\theta_i/2$ (thus choosing an order of the eigenvalues or equivalently, if the reader prefers, a quasi-parabolic structure at each singularity), and suppose A_4 is diagonalisable. Conjugate the system so that

$$A_4 = -(A_1 + A_2 + A_3) = \mathrm{diag}(\theta_4, -\theta_4)/2$$

and note that Schlesinger's equations preserve A_4. Since the top-right matrix entry of A_4 is zero, the top-right matrix entry of

$$z(z-1)(z-t)\sum_1^3 \frac{A_i}{z-a_i} \tag{4}$$

is a degree one polynomial in z. Define $y(t)$ to be the position of its unique zero on the complex z line.

***Theorem -1* (see [24]).** — *If $\mathbf{A}(t)$ satisfies the Schlesinger equations then $y(t)$ satisfies P_{VI}:*

$$\begin{aligned}\frac{d^2y}{dt^2} =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &+ \frac{y(y-1)(y-t)}{2\,t^2(t-1)^2}\left((\theta_4-1)^2 - \frac{\theta_1^2\, t}{y^2} + \frac{\theta_3^2(t-1)}{(y-1)^2} + \frac{(1-\theta_2^2)t(t-1)}{(y-t)^2}\right).\end{aligned}$$

Phrased differently, for each fixed t, the prescription above defines a function y on $\mathcal{O}$, which makes up half of a system of (canonical) coordinates, defined on a dense open subset. A conjugate coordinate x can be explicitly defined and one can write the isomonodromy connection explicitly in the coordinates x, y on $\mathcal{O}$ to obtain a coupled system of first-order nonlinear equations for $x(t), y(t)$ (see [**24**], where our x is denoted $\widetilde{z}$). Then, eliminating x yields the second order equation $\mathrm{P_{VI}}$ for y. (One consequence is that if y solves $\mathrm{P_{VI}}$ there is a direct relation between x and the derivative y', as in equation (6) below.)

In the remainder of this article the main aims are to:

•1) Explain how Okamoto's affine F_4 Weyl group symmetries of $\mathrm{P_{VI}}$ arise from natural symmetries of Schlesinger equations, and

•2) Describe the classification of the Platonic solutions to P_{VI} (i.e., those solutions having linear monodromy group equal to the symmetry group of a Platonic solid).

The key step for •1) (which also led us to •2)) is to use a different realisation of P_{VI}, as controlling isomonodromic deformations of certain 3×3 Fuchsian systems. Note that these results have been written down elsewhere, although the explicit formulae of Remarks 6 and 7 are new and constitute a direct verification of the main results about the 3×3 Fuchsian realisation. Note also that the construction of the Platonic solutions has evolved rapidly recently (e.g., since the author's talk in Angers and since the first version of [**13**] appeared). For example, there are now simple explicit formulae for all the Platonic solutions (something that we had not imagined was possible for a long time[(1)]).

Remark 1. — Let us briefly mention some other possible directions that will not be discussed further here. Firstly, by describing P_{VI} in this way the author is trying to emphasise that P_{VI} is the explicit form of the simplest non-abelian Gauss–Manin connection, in the sense of Simpson [**34**], thereby putting P_{VI} in a very general context (propounded further in [**9**, section 7], especially p. 192). For example, suppose we replace the above family of four-punctured spheres (over B) by a family of projective varieties X over a base S, and choose a complex reductive group G. Then (by the same argument as above), one again has a local system of varieties

$$M_B = \mathrm{Hom}(\pi_1(X_s), G)/G$$

over S and one can pull-back along the Riemann–Hilbert map to obtain a flat connection on the corresponding family M_{DR} of moduli spaces of connections. Simpson proves this connection is again algebraic, and calls it the non-abelian Gauss–Manin connection, since M_B and M_{DR} are two realisations of the first non-abelian cohomology group $H^1(X_s, G)$, the Betti and De Rham realisations.

Also, much of the structure found in the regular (-singular) case may be generalised to the irregular case. For example, as Jimbo–Miwa–Ueno [**25**] showed, one can also consider isomonodromic deformations of (generic) irregular connections on a Riemann surface and obtain explicit deformation equations in the case of $\mathbb{P}^1$. This can also be described in terms of nonlinear connections on moduli spaces and there are natural symplectic structures on the moduli spaces which are preserved by the connections [**9, 7**]. Perhaps most interestingly, one obtains extra deformation parameters in the irregular case (one may vary the 'irregular type' of the linear connections as well as the moduli of the punctured curve). These extra deformation parameters turn out to be related to quantum Weyl groups [**10**].

(1) Mainly because the 18 branch genus one icosahedral solution of [**18**] took 10 pages to write down and we knew quite early on that the largest icosahedral solution had genus seven and 72 branches.