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DYNAMICS OF THE SIXTH PAINLEVÉ EQUATION

by

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Abstract. — The sixth Painlevé equation is hiding extremely rich geometric structures behind its outward appearance. In this article, we give a complete picture of its dynamical nature based on the Riemann-Hilbert approach recently developed by the authors and using various techniques from algebraic geometry.

A large part of the contents can be extended to Garnier systems, while this article is restricted to the original sixth Painlevé equation.

***Résumé* (Dynamique de la sixième équation de Painlevé).** — Malgré une apparente simplicité, l'équation de Painlevé VI cache des structures géométriques très riches. Nous en décrivons les aspects dynamiques en nous appuyant sur l'approche de type Riemann-Hilbert récemment développée par les auteurs et en utilisant différentes techniques issues de la géométrie algébrique.

Une grande partie de ces résultats peut être étendue aux systèmes de Garnier. Toutefois, dans cet article, nous nous limitons au cas de l'équation de Painlevé VI.

1. Introduction

The sixth Painlevé equation $\mathrm{P_{VI}} = \mathrm{P_{VI}}(\kappa)$ is among the six kinds of differential equations that were discovered by Painlevé [**65**] and his student Gambier [**18**] around the turn of the twentieth century. It is a second-order nonlinear ordinary differential

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equation with an independent variable $x \in \mathbb{P}^1 - \{0, 1, \infty\}$ and an unknown function $q = q(x)$,

$$\begin{aligned} q_{xx} &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right) q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right) q_x \\ &\quad + \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2 - \kappa_1^2 \frac{x}{q^2} + \kappa_2^2 \frac{x-1}{(q-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\}, \end{aligned} \tag{1}$$

depending on parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in a 4-dimensional affine space

$$\mathcal{K} = \{\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1\}. \tag{2}$$

This highly nonlinear and seemingly rather ugly equation is only a small visible part of a more substantial entity. The large invisible part has extremely rich geometric structures that are related to symplectic geometry, moduli spaces of stable parabolic connections, moduli spaces of representations of a fundamental group, Riemann-Hilbert correspondence, geometry of cubic surfaces, braid and modular groups, simple isolated singularities and their resolutions of singularities, affine Weyl groups, discrete dynamical systems, and so on. The aim of this survey article is to discuss various aspects of these illuminating structures, giving a complete picture of the sixth Painlevé equation.

Among other features, Painlevé equation is primarily a dynamical system and a dynamical system in general is characterized by two aspects: *laws* and *phenomena*. Mathematically, laws refer to equations that govern the dynamics, symmetries of the system, etc., while phenomena refer to solutions of the equations, (global) behaviors of trajectories, etc. These two aspects often show a sharp contrast. For example, in classical mechanics, the simple laws of Newton create extremely rich and complicated phenomena. The Painlevé dynamics is also in this case, being algebraic in its laws and transcendental in its phenomena (see Table 1).

aspect	contents	nature
laws	equations, symmetry, ...	algebraic
phenomena	solutions, trajectories, ...	transcendental

Table 1. Two aspects of Painlevé equation

For comparison, we should remark that there exists an interesting dynamics whose laws are already transcendental, like a dynamics on a K3 surface recently explored by McMullen [**49**], who showed that the existence of Siegel disks implies the transcendence of the K3 surface.

Generally speaking, the two principal approaches to dynamical systems are perhaps:

(L) Lyapunov's methods, (C) conjugacy methods.

In Lyapunov's methods (L), we examine, control, or confine the behaviors of trajectories by estimating suitable functions called "Lyapunov functions". Main tools of the methods are estimations by inequalities. On the other hand, in the conjugacy methods (C), we try to find a "conjugacy map" that converts the difficult dynamical system we want to study to a more tractable one, to extract informations from the latter, and to send feedback to the former (see §2.2 for more details). Our approach to the Painlevé equation, which we call the *Riemann-Hilbert approach*, falls into this category (C), making use of Riemann-Hilbert correspondence as a conjugacy map between Painlevé flow and isomonodromic flow.

Of course, the Riemann-Hilbert approach is closely related to the isomonodromic approach represented by the classical works of Fuchs [**17**], Schlesinger [**71**], Garnier [**20**], Jimbo, Miwa and Ueno [**37, 38**] and others, but differs from the latter in its definitive employment of the method of conjugacy maps and in its extensive use of a complete solution to the Riemann-Hilbert problem. The Riemann-Hilbert approach *a priori* has a global nature once Riemann-Hilbert correspondence is formulated appropriately, while the isomonodromic approach mostly stands on the infinitesimal point of view and pays little attention to the target space of Riemann-Hilbert correspondence, namely, moduli space of monodromy representations. In the Riemann-Hilbert appraoch, we consciously distinguish the Painlevé flow on the moduli space of stable parabolic connections and the isomonodromic flow on the moduli space of monodromy representations, and build a bridge between them via the Riemann-Hilbert correspondence.

This approach has been explored by Iwasaki [**31, 32, 33, 34**], Hitchin [**24**], Kawai [**40, 41**], Boalch [**5, 6, 7**], Dubrovin and Mazzocco [**14**] and others. Recently it was thoroughly developed by Inaba, Iwasaki and Saito [**29, 28**]. The exposition of this article is largely based on the contents of the last papers. We focus our attention on the original case of P_{VI} with the aim of presenting, for the most basic model, those materials which can be expected to be universal throughout various generalizations. A large part of the contents is extended to Garnier systems, a several-variable version of P_{VI}; see [**29**].

In addtion to the general methods represented by approaches (L) and (C), which are conceivable in general situations, there are also various particular methods applicable to various particular dynamical systems. For example, for the class of dynamical systems that are called completely integrable systems, there exist

(CI) methods for complete integration,

which are characterized by such keywords as τ-functions, bilnear equations, Lax pairs, separations of variables, combinatorics and representation theory, etc. Painlevé equations are usually thought of as a member of this class and many works have been done from this point of view. See Conte [**10**], Noumi [**56**] and the references therein. But we shall not touch on this aspect in this article. Among other things, we wish to lay a sound foundation on the sixth Painlevé equation to such an extent that it can be a basis for the investigations into the *transcendental* nature of $\mathrm{P_{VI}}$. To do so, many things should be done, even within the general framework of dynamical systems, before entering into those subjects which are particular to integrable systems. Therefore the integrable aspects should be discussed later and elsewhere.

Lyapunov-type approaches to Painlevé equations will not at all be discussed in this article. There have been long traditions as well as recent developments of establishing Painlevé property by these methods. We refer to Painlevé [**65**], Hukuhara [**25**] (see Okamoto and Takano [**64**] for a part of these unpublished lectures), Joshi and Kruskal [**39**], Steinmetz [**76**], Shimomura [**72**], Iwasaki, Kimura, Shimomura and Yoshida [**35**], Gromak, Laine and Shimomura [**22**] and the references therein.

The organization of this article is as follows: In Section 2 we introduce a general formalism of dynamical systems and cast $\mathrm{P_{VI}}$ into this framework. We present the Guiding Diagram that encodes major dynamical natures of $\mathrm{P_{VI}}$. Section 3 is devoted to the construction of moduli spaces of stable parabolic connections, which, in the dynamical context, means the construction of phase spaces of $\mathrm{P_{VI}}$. In Section 4, after setting up moduli spaces of monodromy representations, we formulate Riemann-Hilbert correspondence, RH, and settle Riemann-Hilbert problems in suitable ways. In the dynamical context, theses parts correspond to the construction of conjugacy maps. In Section 5 we formulate isomonodromic flows $\mathcal{F}_{\mathrm{IMF}}$ and Painlevé flows $\mathcal{F}_{\mathrm{P_{VI}}}$ in such a manner that RH yields analytic conjugacy from $\mathcal{F}_{\mathrm{P_{VI}}}$ to $\mathcal{F}_{\mathrm{IMF}}$. Section 6 is devoted to the construction of a family of affine cubic surfaces, which enables us to describe all the previous constructions more explicitly. In Section 7 we give a characterization of Bäcklund transformations, namely, the symmetries of $\mathrm{P_{VI}}$, in terms of Riemann-Hilbert correspondence. In Section 8 we describe the nonlinear monodromy or the Poincaré return map of $\mathrm{P_{VI}}$ that extracts the global nature of trajectories of $\mathrm{P_{VI}}$. In Section 9 we characterize the classical components of $\mathrm{P_{VI}}$, called the Riccati flows, in terms of singularities on cubic surfaces and their resolutions of singularities. In Section 10 we construct canonical coordinate systems of moduli spaces (phase spaces) which make it possible to write down the Painlevé dynamics explicitly. This article is closed with a brief summary, especially with the Closing Diagram, in Section 11.

2. Painlevé Equation as a Dynamical System

A complete picture of the sixth Painlevé equation is most clearly described in the framework of dynamical systems, or, more specifically as a time-dependent Hamiltonian system with Painlevé property. So we begin by establishing a general formalism of dynamical systems, based on which we shall develop our whole story.

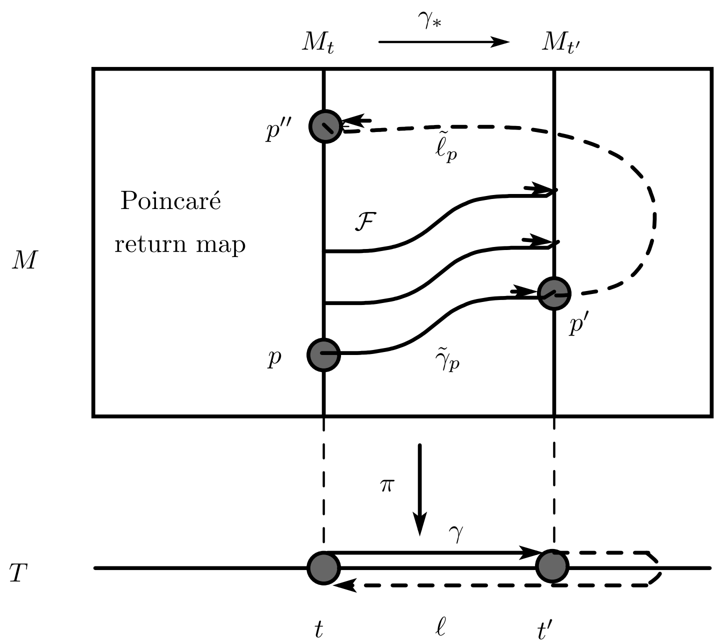

FIGURE 1. Dynamical system with Painlevé property

2.1. General Formalism of Dynamical Systems

***Definition 2.1* (Time-Dependent Dynamical System).** — A *time-dependent dynamical system* $(M, \mathcal{F})$ is a fibration $\pi : M \to T$ of complex manifolds together with a complex foliation $\mathcal{F}$ on M that is transverse to each fiber $M_t = \pi^{-1}(t)$, $t \in T$. The total space M is referred to as the *phase space*, while the base space T is called the *space of time-variables*. Moreover, the fiber M_t is called the *space of initial conditions* at time t.

The space of initial conditions becomes a meaningful concept if the dynamical system enjoys Painlevé property. It is this property that makes it possible to think of Poincaré return maps or the nonlinear monodromy, which is the discrete dynamical system on a space of initial conditions that represents the global nature of a continuous dynamical system.

***Definition 2.2* (Geometric Painlevé Property).** — We say that a dynamical system $(M, \mathcal{F})$ has *geometric Painlevé property* (GPP) if for any path γ in T and any point $p \in M_t$, where t is the initial point of γ, there exists a unique $\mathcal{F}$-horizontal lift $\tilde{\gamma}_p$ of γ with initial point at p (see Figure 1). Here a curve in M is said to be $\mathcal{F}$-horizontal if it lies in a leaf of $\mathcal{F}$.