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POINT CONFIGURATIONS, CREMONA TRANSFORMATIONS AND THE ELLIPTIC DIFFERENCE PAINLEVÉ EQUATION

by

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Abstract. — A theoretical foundation for a generalization of the elliptic difference Painlevé equation to higher dimensions is provided in the framework of birational Weyl group action on the space of point configurations in general position in a projective space. By introducing an elliptic parametrization of point configurations, a realization of the Weyl group is proposed as a group of Cremona transformations containing elliptic functions in the coefficients. For this elliptic Cremona system, a theory of τ-functions is developed to translate it into a system of bilinear equations of Hirota-Miwa type for the τ-functions on the lattice. Application of this approach is also discussed to the elliptic difference Painlevé equation.

Résumé **(Configurations de points, transformations de Cremona et équation de Painlevé aux différences elliptique)**

Dans le cadre de l'action birationnelle du groupe de Weyl sur l'espace des configurations de points en position générale dans un espace projectif on établit des fondements théoriques en vue d'une généralisation aux dimensions supérieures de l'équation de Painlevé aux différences elliptique. On réalise le groupe de Weyl comme un groupe de transformations de Cremona à coefficients fonctions elliptiques grâce à une paramétrisation elliptique des configurations de points. Une théorie des fonctions τ permet de traduire ce système de Cremona en un système d'équations bilinéaires de type Hirota-Miwa pour les fonctions τ sur le réseau. On en donne une application à l'équation de Painlevé aux différences elliptique.

1. Introduction

The main purpose of this paper is to provide a theoretical foundation for a generalization of the elliptic difference Painlevé equation to higher dimensions in the framework of birational Weyl group actions on the spaces of point configurations in general position in projective spaces.

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Since the pioneering work of Grammaticos, Ramani, Papageorgiou and Hietarinta [**5, 19**], discrete Painlevé equations have been studied from various viewpoints. A large class of second order discrete Painlevé equations, as well as their generalizations, has been discovered through the studies of singularity confinement property, bilinear equations, affine Weyl group symmetries and spaces of initial conditions (see [**20, 21, 15, 22**]...). For historical aspects of discrete Painlevé equations, we refer the reader to the review of Grammaticos-Ramani [**4**].

Among many others, we mention here the geometric approach proposed by Sakai [**22**] for a class of discrete Painlevé equations arising from rational surfaces. Each equation in this class is defined by the group of Cremona transformations on a certain family of surfaces obtained from the projective plane $\mathbb{P}^2(\mathbb{C})$ by blowing-up. According to the types of rational surfaces, those discrete Painlevé equations are classified in terms of affine root systems. Also, their symmetries are described by means of affine Weyl groups. The *elliptic difference Painlevé equation*, which is regarded as the master equation for all discrete Painlevé equations of this class, is a discrete dynamical system defined on a family of surfaces parametrized by the 9-point configurations in general position in $\mathbb{P}^2(\mathbb{C})$; the corresponding group of Cremona transformations is the affine Weyl group of type $E_8^{(1)}$. As we have shown in [**8**], this system of difference equations can be transformed into the eight-parameter discrete Painlevé equation of Ohta-Ramani-Grammaticos [**16**], constructed from a completely different viewpoint of bilinear equations for the τ-functions on the E_8 lattice. It is also known by [**8**] that the elliptic difference Painlevé equation has special Riccati type solutions obtained by linearization to the *elliptic difference hypergeometric equation*. This gives a new perspective of nonlinear special functions to the elliptic hypergeometric functions which have been studied for instance by Frenkel-Turaev [**3**] in the context of elliptic 6-j symbols and by Spiridonov-Zhedanov [**23**] in the theory of biorthogonal rational functions on elliptic grids.

Generalizing the geometric approach to the elliptic difference Painlevé equations, in this paper we investigate the configuration space $\mathbb{X}_{m,n}$ of n points $p_1, \dots, p_n$ in general position in the projective space $\mathbb{P}^{m-1}(\mathbb{C})$. It is well-known [**2**] that the Weyl group $W_{m,n}$ associated with the tree $T_{2,m,n-m}$ can be realized as a group of birational transformations on the configuration space $\mathbb{X}_{m,n}$. Through the $W_{m,n}$-equivariant projection $\mathbb{X}_{m,n+1} \to \mathbb{X}_{m,n}$ that maps $[p_1, \dots, p_n, q]$ to $[p_1, \dots, p_n]$, from the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ we obtain a realization of the Weyl group $W_{m,n}$ as a group of Cremona transformations on $q \in \mathbb{P}^{m-1}(\mathbb{C})$ parameterized by the configuration space $\mathbb{X}_{m,n}$. Note that in the case when $(m,n) = (3,9), (4,8)$ or $(6,9)$, the Weyl group $W_{m,n}$ is the affine Weyl group of type $E_8^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$, respectively; this group $W_{m,n} = W(E_l^{(1)})$ decomposes into the semidirect product of the root lattice $Q(E_l)$ and the finite Weyl group $W(E_l)$. In each of the three cases, through the birational action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$, the lattice part of the affine Weyl group provides a *discrete Painlevé system* on $\mathbb{P}^{m-1}(\mathbb{C})$ with parameter space $\mathbb{X}_{m,n}$. The discrete Painlevé system of type $(3,9)$ thus obtained contains the three discrete Painlevé equations, elliptic, trigonometric and rational, with $W(E_8^{(1)})$ symmetry in Sakai's table.

In this framework of configuration spaces, in Section 4 we construct a $W_{m,n}$-equivariant meromorphic mapping $\varphi_{m,n} : \mathfrak{h}_{m,n} \dashrightarrow \mathbb{X}_{m,n}$ by means of elliptic functions, where $\mathfrak{h}_{m,n}$ denotes the Cartan subalgebra of the Kac-Moody Lie algebra associated with the tree $T_{2,m,n-m}$. If we regard the birational $W_{m,n}$-action on $\mathbb{X}_{m,n}$ as a system of functional equations for the coordinate functions, a 'canonical' elliptic solution is provided by the meromorphic mapping $\varphi_{m,n}$. Its image also specifies a $W_{m,n}$-stable class of n-point configurations in $\mathbb{P}^{m-1}(\mathbb{C})$ in which the n points are on an elliptic curve. By restricting the point configurations to this class, from the birational Weyl group action of $W_{m,n}$ on $\mathbb{X}_{m,n+1}$ we obtain a realization of $W_{m,n}$ as a group of Cremona transformations on $\mathbb{P}^{m-1}(\mathbb{C})$ parametrized by elliptic functions, which we call the *elliptic Cremona system* of type (m,n). In Section 5 we develop a theory of τ-functions for this elliptic Cremona system of type (m,n), and show that it is translated into a system of bilinear equations of Hirota-Miwa type for the τ-functions on the lattice. After that we reconsider the case of the elliptic difference Painlevé system of type $(3,9)$ in the scope of the general setting of this paper. There we give explicit description for some of the discrete time evolutions, in terms of homogeneous coordinates in Section 6, and in the language of geometry of plane curves in Section 7.

The τ-function approach developed in this paper can be applied effectively to the study of special hypergeometric solutions of the elliptic Painlevé equation and its degenerations. Also, it is an important problem to *complete* the framework of $\mathbb{X}_{m,n}$ of point configurations in general position, so that it should contain all reasonable degenerate configurations as in Sakai's table. These subjects will be investigated in our subsequent papers.

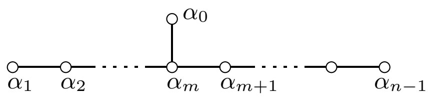
2. Point configurations and Cremona transformations

Let $\mathbb{X}_{m,n}$ be the configuration space of n points in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ $(n > m > 1)$. We say that an n-tuple of points $(p_1, \dots, p_n)$ in $\mathbb{P}^{m-1}(\mathbb{C})$ is *in general position* if $p_1, \dots, p_n$ are mutually distinct, and $\#(H \cap \{p_1, \dots, p_n\}) < m$ for any hyperplane H in $\mathbb{P}^{m-1}(\mathbb{C})$. We denote by $[p_1, \dots, p_n]$ the corresponding *configuration*, namely, the equivalence class of $(p_1, \dots, p_n)$ under the diagonal $PGL_m(\mathbb{C})$-action. By fixing a system of homogeneous coordinates for $\mathbb{P}^{m-1}(\mathbb{C})$, the configuration space $\mathbb{X}_{m,n}$ may be identified with the double coset space

$$\mathbb{X}_{m,n} = GL_m(\mathbb{C}) \backslash \mathrm{Mat}^*_{m,n}(\mathbb{C}) / T_n, \tag{1}$$

where $\mathrm{Mat}^*_{m,n}(\mathbb{C})$ stands for the space of all $m \times n$ complex matrices whose $m \times m$ minor determinants are all nonzero, and $T_n = (\mathbb{C}^*)^n$ for the diagonal subgroup of $GL_n(\mathbb{C})$. The configuration space $\mathbb{X}_{m,n}$ has the structure of an affine algebraic variety, isomorphic to a Zariski open subset of $\mathbb{C}^{(m-1)(n-m-1)}$ (see [**25**], for instance). Also, it is known [**1**], [**2**] that the Weyl group associated with the tree

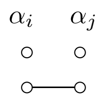
(2) $T_{2,m,n-m}$:



acts birationally on $\mathbb{X}_{m,n}$. This Weyl group $W_{m,n} = W(T_{2,m,n-m})$ is generated by the *simple reflections* $s_0, s_1, \ldots, s_{n-1}$ with the following fundamental relations.

(3) $W_{m,n} = \langle s_0, s_1, \ldots, s_{n-1} \rangle$:

$$s_i^2 = 1$$
$$s_i s_j = s_j s_i$$
$$s_i s_j s_i = s_j s_i s_j$$



As we will recall below, $W_{m,n}$ is realized as a group of birational transformations of $\mathbb{X}_{m,n}$ by the standard Cremona transformations with respect to m points among $p_1, \ldots, p_n$.

Given a set of m points $p_1, \ldots, p_m$ in general position, choose a system of homogeneous coordinates $x = (x_1, \ldots, x_m)$ such that

$$p_1 = (1 : 0 : \ldots : 0), \quad p_2 = (0 : 1 : \ldots : 0), \quad \ldots, \quad p_m = (0 : \ldots : 0 : 1). \tag{4}$$

Then the *standard Cremona transformation* with respect to $(p_1, \ldots, p_m)$ is the birational transformation $p \to \widetilde{p}$ of $\mathbb{P}^{m-1}(\mathbb{C})$ defined by $\widetilde{p} = \big(x_1^{-1} : \ldots : x_m^{-1}\big)$ for any $p = (x_1 : \ldots : x_m)$ with $x_i \neq 0$ $(i = 1, \ldots, m)$. Note that this transformation depends on the choice of homogeneous coordinates, and is determined only up to the action of $(\mathbb{C}^*)^m$. The birational (right) action of $W_{m,n}$ on $\mathbb{X}_{m,n}$ is then defined as follows. Firstly, the symmetric group $\mathfrak{S}_n$ acts on $\mathbb{X}_{m,n}$ by the permutation of n points:

$$[p_1, \ldots, p_n].\sigma = [p_{\sigma(1)}, \ldots, p_{\sigma(n)}] \qquad (\sigma \in \mathfrak{S}_n). \tag{5}$$

The adjacent transpositions $s_j = (j, j+1)$ $(j = 1, \ldots, n-1)$ provide the simple reflections attached to the subdiagram of type A_{n-1} in $T_{2,m,n-m}$. The remaining simple reflection s_0 is given by the (well-defined) birational transformation

$$[p_1, \ldots, p_n].s_0 = [p_1, \ldots, p_m, \widetilde{p}_{m+1}, \ldots, \widetilde{p}_n], \tag{6}$$

in terms of the standard Cremona transformation $p \to \widetilde{p}$ with respect to the first m points $(p_1, \ldots, p_m)$. These birational transformations $s_0, s_1, \ldots, s_{n-1}$ in fact satisfy the fundamental relations for the simple reflections of $W_{m,n}$. We also remark that, for each subset $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ of mutually distinct m indices, the standard Cremona transformation with respect to $(p_{j_1}, \ldots, p_{j_m})$ is determined as $\mathrm{cr}_{j_1,\ldots,j_m} = \sigma s_0 \sigma^{-1} \in W_{m,n}$ by a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma(a) = j_a$ for $a = 1, \ldots, m$.

The right birational action of $W_{m,n}$ on $\mathbb{X}_{m,n}$ induces a left action of $W_{m,n}$ on the field $\mathcal{K}(\mathbb{X}_{m,n})$ of rational functions on $\mathbb{X}_{m,n}$ as a group of automorphisms: For each $\varphi \in \mathcal{K}(\mathbb{X}_{m,n})$ and $w \in W_{m,n}$, we define $w(\varphi) \in \mathcal{K}(\mathbb{X}_{m,n})$ by

$$w(\varphi)([p_1,\dots,p_n]) = \varphi([p_1,\dots,p_n].w) \tag{7}$$

for any generic $[p_1,\dots,p_n] \in \mathbb{X}_{m,n}$. Let us consider the set $\mathcal{U}_{m,n}$ of all matrices $U \in \mathrm{Mat}_{m,n}(\mathbb{C})^*$ of the form

$$U = \begin{pmatrix} 1 & \dots & 0 & 0 & 1 & u_{1,m+2} & \dots & u_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & 1 & u_{m-1,m+2} & \dots & u_{m-1,n} \\ 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}. \tag{8}$$

It is easily shown that each $(GL_m(\mathbb{C}), T_n)$-orbit in $\mathrm{Mat}^*_{m,n}(\mathbb{C})$ intersects with $\mathcal{U}_{m,n}$ at one point. By using this transversal $\mathcal{U}_{m,n} \stackrel{\sim}{\to} GL_m(\mathbb{C})\backslash\mathrm{Mat}_{m,n}(\mathbb{C})^*/T_n$, we identify $\mathbb{X}_{m,n}$ with a Zariski open subset of the affine space $\mathbb{C}^{(m-1)(n-m-1)}$ with canonical coordinates $u = (u_{i,j})_{1\le i\le m-1; m+2\le j\le n}$. Through the isomorphism $\mathcal{K}(\mathbb{X}_{m,n}) \stackrel{\sim}{\to} \mathbb{C}(u)$, the action of $W_{m,n}$ on $\mathcal{K}(\mathbb{X}_{m,n})$ can be described explicitly in terms of the u variables. The following table shows how the simple reflections s_k $(k = 0, 1, \dots, n-1)$ transform the coordinates $u_{i,j}$ $(i = 1,\dots,m-1; j = m+2,\dots,n)$:

$$\begin{aligned}
&k = 0: && s_0(u_{ij}) = \frac{1}{u_{ij}},\\
&k = 1,\dots,m-2: && s_k(u_{ij}) = u_{s_k(i),j},\\
&k = m-1: && s_{m-1}(u_{ij}) = \begin{cases} \dfrac{u_{ij}}{u_{m-1,j}} & (i = 1,\dots,m-2),\\ \dfrac{1}{u_{m-1,j}} & (i = m-1),\end{cases}\\
&k = m: && s_m(u_{ij}) = 1 - u_{ij},\\
&k = m+1: && s_{m+1}(u_{ij}) = \begin{cases} \dfrac{1}{u_{i,m+2}} & (j = m+2),\\ \dfrac{u_{ij}}{u_{i,m+2}} & (j = m+3,\dots,n),\end{cases}\\
&k = m+2,\dots,n-1: && s_k(u_{ij}) = u_{i,s_k(j)},
\end{aligned} \tag{9}$$

where $s_k(i)$ stands for the action of the adjacent transposition $(k, k+1)$ on the index $i \in \{1,\dots,n\}$. From this representation, for each $w \in W_{m,n}$ we obtain a family of rational functions

$$w(u_{i,j}) = S^w_{i,j}(u) \qquad (i = 1,\dots,m-1; j = m+2,\dots,n) \tag{10}$$

in the u variables; these functions satisfy the consistency relations

$$S^1_{i,j}(u) = u_{ij}, \qquad S^{ww'}_{i,j}(u) = S^{w'}_{i,j}(S^w(u)) \tag{11}$$

for any i, j and $w, w' \in W_{m,n}$.