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SPECIAL POLYNOMIALS ASSOCIATED WITH RATIONAL AND ALGEBRAIC SOLUTIONS OF THE PAINLEVÉ EQUATIONS

by

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Abstract. — Rational solutions of the second, third and fourth Painlevé equations ($\mathrm{P_{II}}$–$\mathrm{P_{IV}}$) can be expressed in terms of logarithmic derivatives of special polynomials that are defined through coupled second order, bilinear differential-difference equations which are equivalent to the Toda equation.

In this paper the structure of the roots of these special polynomials, and the special polynomials associated with algebraic solutions of the third and fifth Painlevé equations, is studied and it is shown that these have an intriguing, highly symmetric and regular structure. Further, using the Hamiltonian theory for $\mathrm{P_{II}}$–$\mathrm{P_{IV}}$, it is shown that all these special polynomials, which are defined by differential-difference equations, also satisfy fourth order, bilinear ordinary differential equations.

***Résumé* (Polynômes spéciaux associés aux solutions rationnelles ou algébriques des équations de Painlevé)**

On peut exprimer les solutions rationnelles des équations $\mathrm{P_{II}}$, $\mathrm{P_{III}}$ et $\mathrm{P_{IV}}$ en fonction des dérivées logarithmiques de polynômes spéciaux définis par des équations différences-différentielles bilinéaires d'ordre deux couplées et équivalentes à l'équation de Toda.

Dans cet article nous étudions la configuration des racines de ces polynômes spéciaux et des polynômes spéciaux associés aux solutions algébriques des équations de Painlevé $\mathrm{P_{III}}$ et $\mathrm{P_{V}}$. Nous mettons en évidence une structure étonnante, fortement symétrique et régulière. En outre, appliquant la théorie hamiltonienne à $\mathrm{P_{II}}$, $\mathrm{P_{III}}$ et $\mathrm{P_{IV}}$, nous montrons que tous ces polynômes spéciaux, définis par des équations différences-différentielles, satisfont aussi à des équations différentielles ordinaires bilinéaires d'ordre 4.

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1. Introduction

In this paper our interest is in rational solutions of the second, third and fourth Painlevé equations ($\mathrm{P_{II}}$–$\mathrm{P_{IV}}$)

$$w'' = 2w^3 + zw + \alpha, \tag{1.1}$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{1.2}$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{1.3}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$ and α, β, γ and δ are arbitrary constants and algebraic solutions of $\mathrm{P_{III}}$ and the fifth Painlevé equation ($\mathrm{P_V}$)

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}. \tag{1.4}$$

The six Painlevé equations ($\mathrm{P_I}$–$\mathrm{P_{VI}}$), were discovered by Painlevé, Gambier and their colleagues whilst studying which second order ordinary differential equations of the form

$$w'' = F\left(z, w, w'\right), \tag{1.5}$$

where F is rational in w' and w and analytic in z, have the property that the solutions have no movable branch points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property* (cf. [**34**]). The Painlevé equations can be thought of as nonlinear analogues of the classical special functions. Indeed Iwasaki, Kimura, Shimomura and Yoshida [**35**] characterize the Painlevé equations as "the most important nonlinear ordinary differential equations" and state that "many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions" (see also [**14, 75**]). The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of a new transcendental function to describe their solution (cf. [**34, 75**]).

Although first discovered from strictly mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications including statistical mechanics, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics. Further the Painlevé equations have attracted much interest since they also arise as reductions of the soliton equations which are solvable by inverse scattering (cf. [**1**], and references therein, for further details).

Vorob'ev [**79**] and Yablonskii [**80**] expressed the rational solutions of $\mathrm{P_{II}}$ (1.1) in terms of the logarithmic derivative of certain special polynomials which are now known as the *Yablonskii–Vorob'ev polynomials* (see §2 below). Okamoto [**60**] derived analogous special polynomials related to some of the rational solutions of P_{IV}, these polynomials are now known as the *Okamoto polynomials* (see §4.2 below), which have been generalised by Noumi and Yamada [**58**] so that all rational solutions of P_{IV} can be expressed in terms of the logarithmic derivative of special polynomials (see §4.3 below). Umemura [**77**] derived associated analogous special polynomials with certain rational and algebraic solutions of P_{III}, P_{V} and P_{VI} which have similar properties to the Yablonskii–Vorob'ev polynomials and the Okamoto polynomials (see also [**56, 81**]). Subsequently there have been several studies of special polynomials associated with the rational solutions of P_{II} [**26, 38, 40, 68**], the rational and algebraic solutions of P_{III} [**39, 59**], the rational solutions of P_{IV} [**26, 41, 58**], the rational solutions of P_{V} [**51, 57**] and the algebraic solutions of P_{VI} [**45, 44, 50, 69, 70**]. Many of these papers are concerned with the combinatorial structure and determinant representation of the polynomials, often related to the Hamiltonian structure and affine Weyl symmetries of the Painlevé equations. Typically these polynomials arise as the "τ-functions" for special solutions of the Painlevé equations and are generated through nonlinear, three-term recurrence relations which are Toda-type equations that arise from the associated Bäcklund transformations of the Painlevé equations. Additionally the coefficients of these special polynomials have some interesting, indeed somewhat mysterious, combinatorial properties (cf. [**56, 75, 77**]).

Clarkson and Mansfield [**22**] investigated the locations of the zeroes of the Yablonskii–Vorob'ev polynomials in the complex plane and showed that these zeroes have a very regular, approximately triangular structure (see also [**15**]). An earlier study of the distribution of the zeroes of the Yablonskii–Vorob'ev polynomials is given by Kametaka, Noda, Fukui, and Hirano [**42**] — see also [**35**, p. 255, p. 339]. The structure of the zeroes of the polynomials associated with rational and algebraic solutions of P_{III} is studied in [**17**], which essentially also have an approximately triangular structure, and with rational solutions of P_{IV} in [**16**], which have an approximate rectangular and combinations of approximate rectangular and triangular structures. The term "approximate" is used since the patterns are not exact triangles and rectangles since the zeroes lie on arcs rather than straight lines.

In this paper we review the studies of special polynomials associated with rational solutions of P_{II}, P_{III} and P_{IV} in §§2–4, respectively, and special polynomials associated with algebraic solutions of P_{III} and P_{V} in §5 and §6, respectively. Further we discuss the rational solutions of the Hamiltonian systems associated with P_{II}, P_{III} and P_{IV}, respectively. In particular, it is shown that the associated special polynomials, which are defined by differential-difference equations, also satisfy fourth order, bilinear ordinary differential equations. This is analogous to classical orthogonal polynomials, such as Hermite, Laguerre and Jacobi polynomials, which satisfy linear ordinary differential, difference and differential-difference equations (cf. [**3, 7, 71**]), and so provides further evidence that the Painlevé equations are nonlinear special functions. In §7 we discuss the interlacing of the roots of these special polynomials in the complex plane. In §8 we discuss our results and pose some open questions.

2. Special Polynomials Associated with Rational Solutions of P_{II}

Rational solutions of P_{II}, for $\alpha = n \in \mathbb{Z}$, can be expressed in terms of the logarithmic derivative of special polynomials which are defined through a second order, bilinear differential-difference equation, see equation (2.2) below. These special polynomials were introduced by Vorob'ev [**79**] and Yablonskii [**80**], now known as the *Yablonskii–Vorob'ev polynomials*, which are given in the following theorem (see also [**26, 68, 75, 78**]).

Theorem 2.1. — *Rational solutions of* P_{II} *exist if and only if* $\alpha = n \in \mathbb{Z}$*, which are unique, and have the form*

$$w_n = w(z;n) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{Q_{n-1}(z)}{Q_n(z)}\right]\right\}, \tag{2.1}$$

for $n \geq 1$*, where the polynomials* $Q_n(z)$ *satisfy the differential-difference equation*

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\left[Q_nQ_n'' - (Q_n')^2\right], \tag{2.2}$$

with $Q_0(z) = 1$ *and* $Q_1(z) = z$*. The other rational solutions of* P_{II} *are given by* $w_0 = 0$ *and* $w_{-n} = -w_n$*.*

The Yablonskii–Vorob'ev polynomials $Q_n(z)$ are monic polynomials of degree $\frac{1}{2}n(n+1)$ with integer coefficients. It is clear from the recurrence relation (2.2) that the $Q_n(z)$ are rational functions, though it is not obvious that in fact they are polynomials since one is dividing by $Q_{n-1}(z)$ at every iteration. Hence it is somewhat remarkable that the Yablonskii–Vorob'ev polynomials are polynomials. A list of the first few Yablonskii–Vorob'ev polynomials and plots of the locations of their zeros in the complex plane are given in [**22**]. A plot of the roots of $Q_{25}(z)$ in the complex plane is given in Figure 2. The interlacing of the roots of these special polynomials in the complex plane is discussed in §7.

It is well-known that P_{II} can be written as the Hamiltonian system [**60**]

$$q' = \frac{\partial \mathrm{H}_{\mathrm{II}}}{\partial p} = p - q^2 - \tfrac{1}{2}z, \qquad p' = -\frac{\partial \mathrm{H}_{\mathrm{II}}}{\partial q} = 2qp + \alpha + \tfrac{1}{2}, \tag{2.3}$$

where the (non-autonomous) Hamiltonian $\mathrm{H}_{\mathrm{II}}(q,p,z;\alpha)$ is given by

$$\mathrm{H}_{\mathrm{II}}(q,p,z;\alpha) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q. \tag{2.4}$$

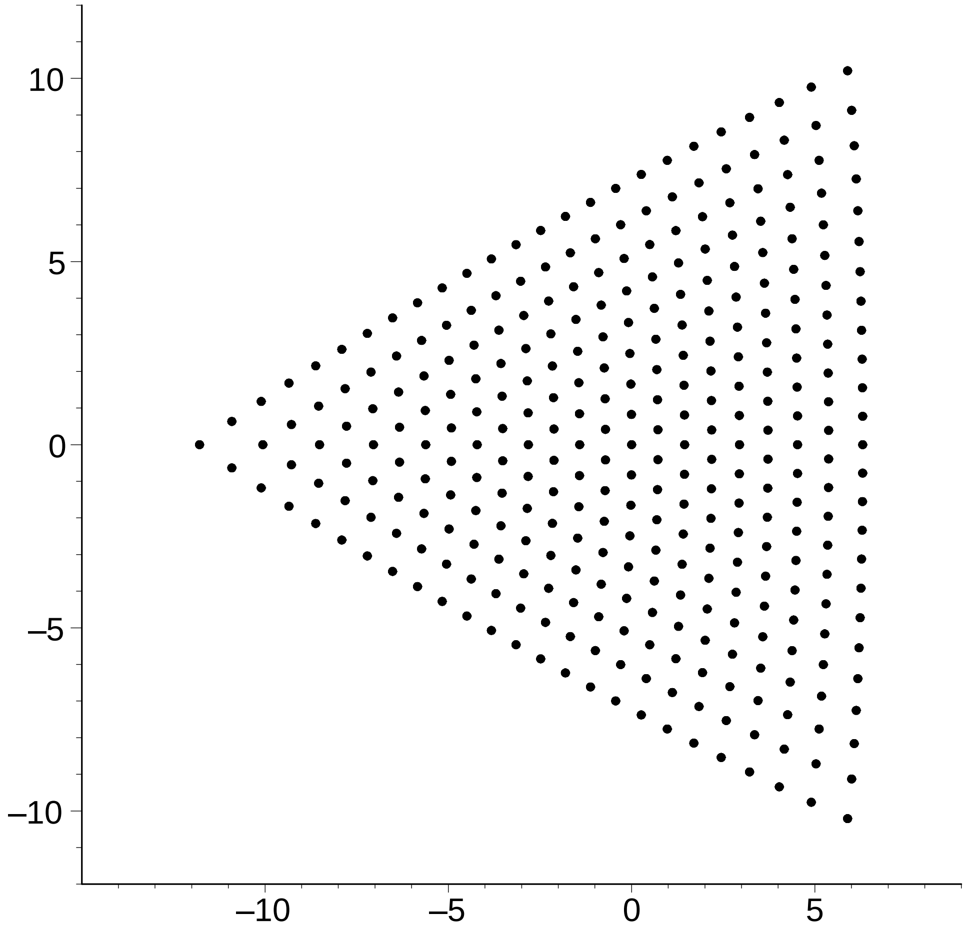

FIGURE 2.1. Roots of the Yablonskii–Vorob'ev polynomial $Q_{25}(z)$

Eliminating p in (2.3) then $q = w$ satisfies $\mathrm{P_{II}}$, whilst eliminating q yields

$$pp'' = \tfrac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 = \tfrac{1}{2}(p')^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2, \tag{2.5}$$

which is known as P_{34}, since it is equivalent to equation XXXIV of Chapter 14 in [**34**]. The Hamiltonian function $\sigma(z;\alpha) = \mathrm{H_{II}}(q, p, z; \alpha)$, where p and q satisfy (2.3), satisfies the second order, second degree equation [**36, 60**]

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2. \tag{2.6}$$