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ON THE ALTERNATE DISCRETE PAINLEVÉ EQUATIONS AND RELATED SYSTEMS

by

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Abstract. — We examine the family of discrete Painlevé equations which were introduced under the qualifier of "alternate". We show that there exists a transformation between the two canonical forms of these equations, and we proceed to link these forms to the contiguity relations of the continuous P_{VI}. We describe the full degeneration cascade of this contiguity, obtaining all related discrete Painlevé equations (among which, one which has never been derived before) as well as mappings which are solvable by linearisation.

Résumé **(Sur les équations discrètes alternatives de Painlevé et les systèmes associés)**
Nous étudions la famille des équations de Painlevé discrètes dites « alternatives ». Nous exhibons une transformation entre les deux formes canoniques et nous relions celles-ci aux relations de contiguïté de l'équation de Painlevé continue P_{VI}. Nous décrivons la cascade de dégénérescence complète liée à cette contiguïté ; nous explicitons toutes les équations de Painlevé discrètes correspondantes (dont une inconnue à ce jour) ainsi que des applications résolubles par linéarisation.

1. Introduction

While the discrete Painlevé equations (d-$\mathbb{P}$s) have properties which mirror those of their continuous counterparts, there exists an aspect where the two families differ drastically: it concerns the abundance of the two sets of equations. The continuous Painlevé equations are traditionally given under six canonical forms [**6**] and, while the situation is somewhat more complicated than that [**2**], it remains that their number is restricted and small. The number of the known discrete Painlevé equations, on the other hand, has been steadily increasing resulting to, literally, dozens of various discrete analogues of the discrete Painlevé transcendental equations [**4**]. To fix the ideas, we remind the reader that the term discrete Painlevé equations is used to designate a nonlinear, nonautonomous, integrable, second order mapping, the continuous limit of

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which is a Painlevé equation. This last feature has been the source of two difficulties. First, as was shown in various works, and proven in a systematic way by Sakai [**15**], the discrete Painlevé equations may contain up to seven parameters while the number of parameters of the continuous Painlevé equations cannot exceed four (in the case of $\mathrm{P_{VI}}$). Thus, all continuous limits of discrete Painlevé equations, with a number of parameters at least equal to four, are constrained to lead to $\mathrm{P_{VI}}$. Second, even for discrete equations with a number of parameters less than four, one has a profusion of equations with the same continuous limit. What is most unfortunate is that, when the discrete Painlevé equations were first discovered, their naming was based essentially on their continuous limit [**13**]. Thus, a d-$\mathbb{P}$ with continuous limit $\mathrm{P_I}$ was called "discrete $\mathrm{P_I}$" and so on. These difficulties were alleviated later, thanks to the Sakai classification, based on the affine Weyl group that describes the transformations of each d-$\mathbb{P}$. But the traditional names of the d-$\mathbb{P}$s, once introduced, turned out to be impossible to eradicate. Among the various ways to deal with the nomenclature difficulty was the introduction of qualifiers like "standard", "alternate", "asymmetric" and so on.

Thus, when in [**3**] we derived the d-$\mathbb{P}$:

$$\frac{z_{n-1}+z_n}{1-x_{n-1}x_n}+\frac{z_n+z_{n+1}}{1-x_nx_{n+1}}=x_n+\frac{1}{x_n}+2z_n+2\mu \tag{1.1}$$

and found that its continuous limit was $\mathrm{P_{II}}$, we dubbed it alternate d-$\mathrm{P_{II}}$ ("alternative" could have been a better choice of adjective), in order to distinguish it from the "standard" d-$\mathrm{P_{II}}$, $x_{n+1}+x_{n-1}=(z_nx_n+a)/(1-x_n^2)$. In the same paper, the alternate d-$\mathrm{P_I}$ was also obtained:

$$\frac{z_{n-1}+z_n}{x_{n-1}+x_n}+\frac{z_n+z_{n+1}}{x_n+x_{n+1}}=x_n^2+1 \tag{1.2}$$

As a matter of fact, (1.2) is the difference equation obtained by Jimbo and Miwa in [**7**] from the contiguity relations of the solutions of the (continuous) Painlevé II.

The alternate d-$\mathrm{P_{II}}$ has been the object of a very detailed study [**8**], where we have presented its Lax pair, Miura transformations, auto-Bäcklund transformations, special solutions, and so on. Moreover, the study of alternate d-$\mathrm{P_{II}}$ has revealed the property of self-duality, which has been crucial for the geometrical description of discrete $\mathbb{P}$s in terms of affine Weyl groups.

In this paper, we examine the equations of the "alternate" family, from a slightly different point of view. We show in particular how they can be derived from the contiguity relations of the solutions of the continuous $\mathrm{P_{VI}}$ equation. While the systematic application of this approach mostly leads to known d-$\mathbb{P}$s, we obtain also one new d-$\mathbb{P}$ which has an unusual form. We show how we can, starting from the contiguity relation, obtain also discrete equations which are not d-$\mathbb{P}$s but linearisable mappings.

2. The canonical form of alternate d-$\mathbb{P}$s

Finding the canonical form of a given equation, be it differential or difference, is a highly nontrivial task. The criteria of "canonicity" are not always explicit and thus, sometimes, the choice of the canonical form is a question of ... choice. For the discrete $\mathbb{P}$s, both difference- and q-, we have presented in [**12**] an approach which classified the forms based on the QRT matrices of the mapping that one finds in the autonomous limit of the d-$\mathbb{P}$. However, this approach concerns only what we have called the "standard" family of d-$\mathbb{P}$s, and thus does not apply to the alternate forms.

For d-$\mathbb{P}$s, the only transformations that are allowed in order to bring a given d-$\mathbb{P}$ under canonical form are homographic transformations. For reasons that will become obvious in the next section, we have sought a transformation which would bring the l.h.s. of the alternate d-$\mathrm{P_{II}}$ equation to the form of the l.h.s. of the alternate d-$\mathrm{P_I}$. This transformation turns out to be simply:

$$y = \frac{x+1}{x-1} \tag{2.1}$$

Thus, starting from (1.1) we obtain the mapping:

$$\frac{z_{n-1}+z_n}{y_{n-1}+y_n} + \frac{z_n+z_{n+1}}{y_n+y_{n+1}} = \frac{4(z_n y_n+\mu)}{y_n^2-1} + \frac{4(y_n^2+1)}{(y_n^2-1)^2} \tag{2.2}$$

While the l.h.s. of the equation becomes identical to that of alternate d-$\mathrm{P_I}$, the r.h.s. becomes substantially more complicated.

At this point, it is interesting to notice that the transformation (2.1) is an involution, and therefore it transforms the l.h.s. of the alternate d-$\mathrm{P_{II}}$ into that of the alternate d-$\mathrm{P_I}$ and *vice-versa*. Indeed applying the transformation (2.1) to (1.2) we obtain:

$$\frac{z_{n-1}+z_n}{1-y_{n-1}y_n} + \frac{z_n+z_{n+1}}{1-y_n y_{n+1}} = \frac{4z}{1-y_n} + \frac{4y_n(y_n^2+1)}{(1-y_n)^4} \tag{2.3}$$

Finally, we wish to point out another interesting transformation that exists for equations of the form of alternate d-$\mathrm{P_I}$ and alternate d-$\mathrm{P_{II}}$. It consists in simply inverting x. For an equation of the form (1.1) and r.h.s. $R(x)$ we find that, after the transformations that restore the l.h.s. to its initial form, the r.h.s. becomes $R'(x) = 4z - R(\frac{1}{x})$. In particular, for the alternate d-$\mathrm{P_{II}}$ we find that the equation is invariant if we invert x provided we change the sign of x and μ. In the alternate d-$\mathrm{P_I}$ case, if we start from an equation of the form (1.2) and r.h.s. $R(x)$, we obtain, after the proper manipulations so as to leave the l.h.s. invariant, a new r.h.s. $R'(x) = \frac{4z}{x} - \frac{1}{x^2}R(\frac{1}{x})$.

The transformations presented in this section do show that there is no reason to prefer an alternate d-$\mathrm{P_I}$ form to an alternate d-$\mathrm{P_{II}}$ one, and vice-versa. Still, they cannot settle the question of finding the canonical form of the alternate d-$\mathbb{P}$s. In order to provide a satisfactory answer we must go back to the origin of these equations. As we have shown in [**3**], these d-$\mathbb{P}$s stem from contiguity relations of the continuous $\mathrm{P_{II}}$

and $P_{\rm III}$ respectively. Thus, the question of canonical forms of the alternate d-$\mathbb{P}$s can be recast as a question on the canonical form of contiguity relations of continuous $\mathbb{P}$s. This makes possible to enlarge the scope of the investigations and analyse in more generality the discrete $\mathbb{P}$s that appear as contiguities.

3. Discrete $\mathbb{P}$s as contiguity relations of continuous $\mathbb{P}$s

Since we are going to examine the relations of discrete to continuous $\mathbb{P}$s through the contiguities of the latter, it is natural to start with the most general continuous $\mathbb{P}$, namely $P_{\rm VI}$. The discrete $\mathbb{P}$s related to $P_{\rm VI}$ have been examined already in [**10**], and also in [**9**]. In what follows, we shall present a somewhat different approach which will allow the treatment of the equations involved on the same footing.

The continuous $P_{\rm VI}$ equation is traditionally given in the form

$$
\begin{aligned}
w'' &= \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)w'^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)w' \\
&\quad +\frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left(\alpha^2-\frac{\beta^2}{w^2}+\frac{\gamma^2(t-1)}{(w-1)^2}+\frac{(1-\delta^2)t(t-1)}{(w-t)^2}\right)
\end{aligned}
\tag{3.1}
$$

where in this particular parametrisation α, β, γ and δ are exactly the monodromy exponents θ_∞, θ_0, θ_1 and θ_t. In this form it is assumed that three of the singular points of $P_{\rm VI}$ are located at the fixed points $\infty, 0, 1$, while the last singular point at t remains movable. While this is an assumption that simplifies substantially the form of $P_{\rm VI}$, it is by no means necessary. It is in fact interesting to investigate the form of $P_{\rm VI}$ when all four singular points are movable: at $a(t)$, $b(t)$, $c(t)$ and $d(t)$. In this case, we can rewrite $P_{\rm VI}$ as (with the constraint $\frac{(a-d)(b-c)}{(a-c)(b-d)}=t$)

$$
\begin{aligned}
w'' &= \frac{1}{2}\Big(\frac{1}{w-a}+\frac{1}{w-b}+\frac{1}{w-c}+\frac{1}{w-d}\Big)w'^2 \\
&\quad -\Big(\frac{1}{t}+\frac{1}{t-1}+\frac{a'}{w-a}+\frac{b'}{w-b}+\frac{c'}{w-c}+\frac{d'}{w-d}+e\Big)w' \\
&\quad +(w-a)(w-b)(w-c)(w-d) \\
&\qquad \Big(\frac{f}{(w-a)^2}+\frac{g}{(w-b)^2}+\frac{h}{(w-c)^2}+\frac{k}{(w-d)^2}\Big)
\end{aligned}
\tag{3.2}
$$

where

$$
e=\frac{a'-b'}{a-b}+\frac{a'-c'}{a-c}+\frac{(a-d)(b-c)}{(a-b)(a-c)}+\frac{(a-d)a'}{(a-b)(a-c)}+\frac{(a-d)b'}{(a-b)(c-b)}+\frac{(a-d)c'}{(a-c)(b-c)}
$$

and f, g, h, k are lengthy expressions which cannot be given in a paper of reasonable length. They are of the form $f = f_0\alpha^2 + f_1$, and similarly for g, h, k, where the derivatives of a, b, c, d appear only in f_1, g_1, h_1, k_1. In order to recover the "standard" expression (3.1) we take $a \to \infty$, $b \to 0$, and $c \to 1$, whereupon $d \to t$.

The Miura transformations associated to (3.1) have been derived in the form of a first degree relation by Okamoto [**11**] and rediscovered in a different approach by Nijhoff and collaborators [**9**]. Using this Miura transformation one can derive the Schlesinger transformations for $\mathrm{P_{VI}}$ and obtain the contiguity relations following the well-established procedure [**3**]. (We will avoid at this point all comments on the fine distinctions on the "Schlesinger transformation" proper terminology [**1**]. While these distinctions are necessary when one aims at a rigorous treatment they are beyond the scope of the practical approach adopted here, where our aim is just the derivation of discrete $\mathbb{P}$s).

We shall not go into the details of the derivation of the contiguity relation of $\mathrm{P_{VI}}$. They can essentially be found in [**10**] and [**1**]. Adopting the convenient form of the latter (and correcting a factor of 2 misprint) we can rewrite the contiguity relation as

$$\begin{aligned}\frac{z_{n-1}+z_n}{x_n-x_{n-1}}+\frac{z_n+z_{n+1}}{x_n-x_{n+1}}&=\\ &\frac{z_n+p(-1)^n}{x_n-a}+\frac{z_n+q(-1)^n}{x_n-b}+\frac{z_n+r(-1)^n}{x_n-c}+\frac{z_n+s(-1)^n}{x_n-d}\end{aligned}\tag{3.3}$$

where $z_n=\delta(n-n_0)$ and we have the constraint $p+q+r+s{=}0$. Expression (3.3) is the contiguity relation for a general position of the singularities corresponding to equation (3.2). Notice that if one of the singular points, say a, is taken to ∞, then the r.h.s. has only three terms and no constraint exists between the surviving q,r,s. Bringing the positions of the singularities in (3.3) to the "standard" ones ∞, 0, 1, t, involves homographic transformations of the independent variables, which amounts to going backwards from (3.2) to (3.1).

A more interesting transformation one can perform on (3.3) is to treat the even- and odd-index xs in a different way. For example, if we reverse the sign of one x out of two, i.e., $x_n\to(-1)^n x_n$, then we obtain an equation (which is reminiscent of that of alternate d-$\mathrm{P_I}$):

$$\frac{z_{n-1}+z_n}{x_n+x_{n-1}}+\frac{z_n+z_{n+1}}{x_n+x_{n+1}}=\sum_{i=1,4}\frac{z_n-\alpha_i(-1)^n}{x_n-a_i(-1)^n}\tag{3.4}$$

Similarly, if we invert one x out of two, i.e. $x_n\to x_n^{(-1)^n}$, we obtain a form reminiscent of alternate d-$\mathrm{P_{II}}$:

$$\frac{z_{n-1}+z_n}{1-x_nx_{n-1}}+\frac{z_n+z_{n+1}}{1-x_nx_{n+1}}=\sum_{i=1,4}\frac{z_n-\alpha_i(-1)^n}{1-x_na_i^{(-1)^{n+1}}}\tag{3.5}$$

This mapping is just the one obtained in [**10**], with specific values for the a_is, precisely as a contiguity of the solutions of $\mathrm{P_{VI}}$ (but, also, initially as a similarity reduction of the discrete mKdV).

Now, that the general framework is set, we turn to the cases obtained from (3.3) or equivalently, (3.4) or (3.5), by degeneration through coalescence of singularities.