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GALOIS THEORY AND PAINLEVÉ EQUATIONS

by

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Abstract. — The paper consists of two parts. In the first part, we explain an excellent idea, due to mathematicians of the 19-th century, of naturally developing classical Galois theory of algebraic equations to an infinite dimensional Galois theory of non-linear differential equations. We show with an instructive example how we can realize the idea of the 19-th century in a rigorous framework. In the second part, we ask questions arising from general Galois theory and Galois theoretic study of Painlevé equations. We also propose an infinite dimensional Galois theory of difference equations.

***Résumé* (Théorie de Galois et Équations de Painlevé).** — Dans une première partie, nous rappelons une excellente idée de mathématiciens du $19^{\text{ème}}$ siècle en vue d'étendre la théorie de Galois classique pour les équations algébriques en une théorie de Galois de dimension infinie pour les équations différentielles non-linéaires. Nous illustrons par un exemple instructif comment concrétiser cette idée de façon rigoureuse.

Dans une deuxième partie, nous formulons des questions liées à la théorie de Galois générale et aux aspects galoisiens des équations de Painlevé. Nous esquissons, en outre, une théorie de Galois de dimension infinie pour les équations aux différences.

1. Introduction

Since Lie tried to apply the rich idea of Galois and Abel in algebraic equations to analysis, Galois theory of differential equations has been attracting mathematicians. Finite dimensional differential Galois theory was developed by Picard, Vessiot and Kolchin and is widely accepted. As Lie already noticed it, the most important part of differential Galois theory is, however, infinite dimensional. After a few trails have been done about 100 years ago, the subject was almost forgotten. We proposed a differential Galois theory of infinite dimension [**14**] in 1996 which is a Galois theory

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of differential field extension. On the other hand, a Galois theory of foliation by B. Malgrange [**11**] that is also infinite dimensional, appeared in 2001. We do not feel that they are well understood.

Our aim in Part I, Invitation to Galois theory, is to explain with examples that our theory is a consequence of natural development of Galois theory of algebraic equations. We recall how mathematicians of the 19-th century understood Galois theory of algebraic equations and extend it to linear ordinary differential equations in §§2 and 3. §4 is the most substantial section of the first part. We show a marvelous idea of mathematicians of the 19-th century in Subsection 4.1 and realize it in the framework of algebraic geometry. Since the reader can find rigorous reasonings in [**14**], we repeatedly use a concrete and yet sufficiently general case, Instructive Case (IC) in Subsection 4.4, to illustrate clearly what is going on.

In Part II, we ask questions about (1) general Galois theory and (2) Galois theoretic study of Painlevé equations. Among the questions about general Galois theory, we cite descent of the field of definition of our Galois group Infgal(L/K) (Questions 1, 2 and 3) and comparison of Malgrange's theory and ours (Question 4), while calculation of Galois group of Painlevé equations (Question 6), understanding of a remarkable paper of Drach on the sixth Painlevé equation (Questions 7, 8, ..., 11) and arithmetic property of the sixth Painevé equation (Questions 17 and 18) belong to the questions about Galois theoretic study of Painlevé equations. We also propose a Galois theory of difference equation of infinite dimension and calculation of Galois group for qP6 of Jimbo and Sakai (Question 12). We added a star to those questions that seem to require a new idea. The mark is, however, nothing more than a personal impression of the author.

PART I
INVITATION TO GALOIS THEORY

2. Galois theory of algebraic equations

The aim of the first part is to explain how an intuitive idea of Galois theory of algebraic equations develops to infinite dimensional differential Galois theory of non-linear differential equations. We described the latter rigorously in a general framework [**14**]. In this note we try to be more intuitive than formal so that the reader can realize how natural the basic idea of our theory is.

Principal homogeneous space is one of the main ingredients of Galois theory. Let us start by recalling the definition.

Definition 2.1. — *Let G be a group operating on a set S. Then we say that the operation (G, S) is a principal homogeneous space if for an element $s \in S$, the map*

$$G \longrightarrow S, \quad g \longmapsto gs$$

is bijective.

Inspired by Galois theory for algebraic equations, S. Lie had a plan to apply the rich idea of Galois and Abel to differential equations. Galois theory of algebraic equations is an ideal theory and it has been the model of generalizations. Let us go back to the 19-th century and see how the mathematicians of that time understood Galois theory and how they tried to generalize it.

Let K be a field and let

$$F(x) := a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad a_i \in K, \text{ for } 0 \leq i \leq n \tag{1}$$

$a_0 \neq 0$, be an algebraic equation with coefficients in K. We suppose for simplicity the field K is of characteristic 0. We assume that the roots of the algebraic equation (1) are distinct. Then the symmetric group S_n of degree n on the n letters

$$\{1, 2, \cdots, n\}$$

operates on the set

$$S := \{(x_1, x_2, \cdots, x_n) \,|\, F(x_i) = 0, \text{ for } 1 \leq i \leq n, \, x_i \neq x_j \text{ if } i \neq j\}$$

of ordered sets $(x_1, x_2, \cdots x_n)$ of roots as permutations of the roots and

$$(S_n, S)$$

is a principal homogeneous space.

The basic symmetric functions are expressed by coefficients.

$$\sum_{i=1}^{n} x_i = -\frac{a_1}{a_0},$$

$$\sum_{1 \leq i < j \leq n} x_ix_j = \frac{a_2}{a_0},$$

$$\cdots$$

$$x_1x_2 \cdots x_n = (-1)^n \frac{a_n}{a_0}.$$

If there is no constraints among the roots

$$x_1, x_2, \cdots, x_n$$

with coefficients in K other than those that are a consequence of the relations above, then the Galois group of equation (1) is the full symmetric group S_n. If there are constraints, they determine a subgroup G of S_n, consisting of those elements leaving all the constraints invariant, as Galois group of the algebraic equation (1). To be more precise, let us consider all rational functions

$$A_\alpha(X_1,\, X_2, \cdots, X_n) \in K(X_1,\, X_2, \cdots, X_n)$$

of variables $X_1,\, X_2, \cdots, X_n$ with coefficients in K indexed by an appropriate set I such that

$$A_\alpha(x_1,\, x_2, \cdots, x_n) \in K,$$

The constraints $A_\alpha(x)$ determine the Galois group G as a subgroup of the symmetric group S_n consisting of elements of S_n leaving all the constraints $A_\alpha(x)$ invariant. Namely

$$G := \{g \in S_n \mid A_\alpha(x_{g(1)},\, x_{g(2)}, \cdots, x_{g(n)}) = A_\alpha(x_1,\, x_2, \cdots, x_n) \text{ for all } \alpha \in I\}$$

Let us illustrate this by an example. Let us consider the following algebraic equation over $\mathbf{Q}$.

$$x^3 - 7x + 7 = (x - x_1)(x - x_2)(x - x_3) = 0. \tag{2}$$

Upon setting

$$\mathbf{x} = (x_1, x_2, x_3),$$

we have a constraint

$$D(\mathbf{x}) := (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = \pm 7 \in \mathbb{Q}.$$

$D(\mathbf{x})$ takes value $+7$ or -7 according as the order of the roots. In fact, $D(\mathbf{x})^2$ is, by definition, the discriminant of the cubic equation (2) so that

$$D(\mathbf{x})^2 = -4 \times (-7)^3 - 27 \times 7^2 = 49.$$

Indeed, the discriminant of a cubic equation

$$x^3 + ax + b = 0$$

is equal to

$$-4a^3 - 27b^2.$$

The Galois group must leave the constraint

$$D(\mathbf{x}) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

invariant so that the Galois group is a subgroup of the alternating group $A_3 \subset S_3$. We can moreover show that the Galois group coincides with the alternating group A_3.

We see how principal homogeneous spaces appear in this context. To this end, let us set

$$\begin{aligned} S &:= \{(x_1, x_2, x_3) \,|\, F(x_i) = 0\}, \\ S_+ &:= \{\mathbf{x} \in S \,|\, D(\mathbf{x}) = 7\}, \\ S_- &:= \{\mathbf{x} \in S \,|\, D(\mathbf{x}) = -7\} \end{aligned}$$

so that we have

$$S = S_+ \amalg S_-.$$

The alternating group A_3 operates on both sets S_+, S_- and

$$(A_3, S_+), \qquad (A_3, S_-)$$

are principal homogeneous spaces. We started from the principal homogeneous space

$$(S_3, S)$$

and we decompose it to get two principal homogeneous spaces

$$(A_3, S_+), \qquad (A_3, S_-).$$

What makes Galois theory of algebraic equation useful is the fact that we have the Galois correspondence. Let us come back to the algebraic equation (1). We denote by $\bar{K}$ an algebraic closure of K. Let L be a subfield of $\bar{K}$ generated over K by all the roots x_i's for $1 \leq i \leq n$ of the algebraic equation (1). Namely

$$L := K(x_1,\, x_2, \cdots, x_n) \subset \bar{K}.$$

This type of field extension, a field extension generated over a field K by all the roots of an algebraic equation with coefficients in K, is called a Galois extension. Let us denote the Galois group of the equation (1) by $G(L/K)$. We can show that the group $G\,(L/K)$ is isomorphic to the group $\mathrm{Aut}(L/K)$ of K-automorphisms of the field L so that the group G depends only on the field extension L/K that the algebraic equation (1) determines! We owe this eminent idea to Dedekind. Let M be an intermediate field of the field extension L/K. Then since the coefficients of the algebraic equation (1) are in K and hence in M and since

$$L = K(x_1,\, x_2, \cdots, x_n) = M(x_1,\, x_2, \cdots, x_n),$$

the field extension L/M is also Galois. Hence we can speak of the Galois group $G\,(L/M)$ of the field extension L/M, which is a subgroup of the Galois group $G\,(L/K)$. We have thus defined a map φ from the set

$$\mathit{Field}\,(L/K)$$

of intermediate fields of the field extension L/K to the set of subgroups

$$\mathit{Group}\,(G)$$

of the Galois group $G = G\,(L/K)$ sending an intermediate subfield M to the subgroup $G\,(L/M)$:

$$\varphi : \mathit{Field}\,(L/K) \to \mathit{Group}\,(G).$$

Conversely let H be a subgroup of the Galois group $G = G\,(L/K)$. Then H determines an intermediate field

$$L^H := \{z \in L \mid g(z) = z \text{ for every element } g \in H \subset G = \mathrm{Aut}(L/K)\ \}$$

consisting of those elements of the field L that are left invariant by all the element of H that is a subgroup of the field automorphism group $\mathrm{Aut}(L/K)$.