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THE LAX PAIR FOR THE MKDV HIERARCHY

by

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Abstract. — In this paper we give an algorithmic method of deriving the Lax pair for the modified Korteweg-de Vries hierarchy. For each n, the compatibility condition gives the n-th member of the hierarchy, rather than its derivative. A direct consequence of this is that we obtain the isomonodromy problem for the second Painlevé hierarchy, which is derived through a scaling reduction.

***Résumé* (La paire de Lax de la hiérarchie mKdV).** — Dans cet article, nous présentons une méthode algorithmique pour le calcul de la paire de Lax de la hiérarchie de Korteweg-de Vries modifiée. Pour tout n, la condition de compatibilité fournit le $n^{\text{ième}}$ membre de la hiérarchie lui-même et non pas sa dérivée. Grâce à une réduction par l'action du groupe de similarité, nous en déduisons un problème d'isomonodromie pour la deuxième hiérarchie de Painlevé.

1. Introduction

There has been considerable interest in partial differential equations solvable by inverse scattering, the so-called *soliton equations*, since the discovery in 1967 by Gardner, Greene, Kruskal and Miura [**8**] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

In the inverse scattering method, which can be thought of as a nonlinear analogue of the Fourier transform method for linear partial differential equations, the nonlinear

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PDE is expressed as the compatibility of two **linear** equations (the celebrated Lax Pair). Typically, this has the form

$$\Phi_x = \mathcal{L}\Phi, \tag{2}$$

$$\Phi_t = \mathcal{M}\Phi, \tag{3}$$

where Φ is a vector, an eigenfunction, and $\mathcal{L}$ and $\mathcal{M}$ are matrices whose entries depend on the solution $u(x,t)$ of the associated nonlinear partial differential equation. Given suitable initial data $u(x,0)$, one obtains the associated scattering data $S(0)$ by solving the spectral problem (2). The scattering data $S(t)$ is then obtained by solving the temporal problem (3), and finally the solution $u(x,t)$ of the partial differential equation is obtained by solving an inverse problem, which is usually expressed as a Riemann-Hilbert problem and frequently the most difficult part (see, for example, [**1, 4**] and the references therein).

Solutions of the modified Korteweg-de Vries (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{4}$$

are related to solutions of the KdV equation (1) through the Miura transformation $u = v_x - v^2$ [**19**].

Soliton equations all seem to possess several remarkable properties in common including, the "elastic" interaction of solitary waves, i.e. multi-soliton solutions, Bäcklund transformations, an infinite number of independent conservation laws, a complete set of action-angle variables, an underlying Hamiltonian formulation, a Lax representation, a bilinear representation à la Hirota, the Painlevé property, an associated linear eigenvalue problem whose eigenvalues are constants of the motion, and an infinite family of equations, the so-called *hierarchy*, which is our main interest in this manuscript (cf. [**1, 4**]).

The standard procedure for generating the mKdV hierarchy is to use a combination of the Lenard recursion operator for the KdV hierarchy and the Miura transformation, as we shall briefly explain now.

The KdV hierarchy is given by

$$u_{t_{n+1}} + \frac{\partial}{\partial x}\mathcal{L}_{n+1}[u] = 0, \qquad n = 0, 1, 2, \ldots, \tag{5}$$

where $\mathcal{L}_n$ satisfies the Lenard recursion relation [**15**]

$$\frac{\partial}{\partial x}\mathcal{L}_{n+1} = \left(\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x\right)\mathcal{L}_n. \tag{6}$$

Beginning with $\mathcal{L}_0[u] = \frac{1}{2}$, this gives

$$\mathcal{L}_1[u] = u, \qquad \mathcal{L}_2[u] = u_{xx} + 3u^2, \qquad \mathcal{L}_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3,$$

and so on. The first four members of the KdV hierarchy are

$$\begin{aligned}
&u_{t_1} + u_x = 0,\\
&u_{t_2} + u_{xxx} + 6uu_x = 0,\\
&u_{t_3} + u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = 0,\\
&u_{t_4} + u_{xxxxxxx} + 14uu_{xxxxx} + 42u_xu_{xxxx} + 70u_{xx}u_{xxx}\\
&\qquad\qquad + 70u^2u_{xxx} + 280uu_xu_{xx} + 70u_x^3 + 140u^3u_x = 0.
\end{aligned}$$

The mKdV hierarchy is obtained from the KdV hierarchy via the Miura transformation $u = v_x - v^2$ (see [**3, 5, 7**]) and can be written as

$$v_{t_{n+1}} + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_n\left[v_x - v^2\right] = 0, \qquad n = 1, 2, 3, \ldots \tag{7}$$

The first three members of the mKdV hierarchy are

$$\begin{aligned}
&v_{t_1} + v_{xxx} - 6v^2v_x = 0,\\
&v_{t_2} + v_{xxxxx} - 10v^2v_{xxx} - 40v_xv_{xx} - 10v_x^3 + 30v^4v_x = 0,\\
&v_{t_3} + v_{xxxxxxx} - 14v^2v_{xxxxx} - 84vv_xv_{xxxx} - 140vv_{xx}v_{xxx}\\
&\qquad - 126v_x^2v_{xxx} - 182v_xv_{xx}^2 + 70v^4v_{xxx} + 560v^3v_xv_{xx}\\
&\qquad\qquad + 420v^2v_x^3 - 140v^6v_x = 0.
\end{aligned}$$

This procedure generates the mKdV hierarchy. We show how to derive a Lax pair for this hierarchy from the one of the KdV hierarchy in the appendix of this paper. However, this procedure gives rise to a hierarchy which is the derivative of the mKdV hierarchy.

Our interest is in the mKdV hierarchy rather than its derivative. We overcome this by generating the Lax pair for the (undifferentiated) mKdV hierarchy in a straightforward, algorithmic way, by using the AKNS expansion technique [**2**]. We call the result the *natural* Lax pair for the mKdV hierarchy. A direct consequence of this is that we also obtain the isomonodromic problem for the second Painlevé hierarchy. Our natural Lax pair for the mKdV hierarchy yields a natural isomonodromy problem that contains the Flaschka-Newell linear problem as the $n = 1$ case.

We derive the natural Lax pair for the mKdV hierarchy in §2 and the natural isomonodromy problem for the second Painlevé hierarchy in §3. In §4 we discuss our results. The Lax pair arising from that for the KdV hierarchy is derived in the appendix.

2. The Natural Lax Pair for the mKdV Hierarchy

The well known Lax pair for the mKdV equation is

$$\text{(8a)} \qquad \frac{\partial\Phi}{\partial x} = \mathcal{L}\Phi = \begin{pmatrix} -i\zeta & v \\ v & i\zeta \end{pmatrix}\Phi$$

$$\text{(8b)} \qquad \begin{aligned}\frac{\partial\Phi}{\partial t} &= \mathcal{M}\Phi \\ &= \begin{pmatrix} -4i\zeta^3 - 2i\zeta v^2 & 4\zeta^2 v + 2i\zeta v_x - v_{xx} + 2v^3 \\ 4\zeta^2 v - 2i\zeta v_x - v_{xx} + 2v^3 & 4i\zeta^3 + 2i\zeta v^2 \end{pmatrix}\Phi\end{aligned}$$

This Lax pair was first given by Ablowitz, Kaup, Newell, and Segur (AKNS) [**2**]. In the same paper it is suggested that higher order equations in the mKdV hierarchy could be generated by considering higher degree expansions in the entries of $\mathcal{M}$. We follow this procedure here.

Proposition 1. — *For each integer $n \geq 1$, the Lax pair for that n-th equation (7) of the mKdV hierarchy is*

$$\text{(9a)} \qquad \frac{\partial\Phi}{\partial x} = \mathcal{L}\Phi = \begin{pmatrix} -i\zeta & v \\ v & i\zeta \end{pmatrix}\Phi$$

$$\text{(9b)} \qquad \frac{\partial\Phi}{\partial t_{n+1}} = \mathcal{M}\Phi = \begin{pmatrix} \displaystyle\sum_{j=0}^{2n+1} A_j(i\zeta)^j & \displaystyle\sum_{j=0}^{2n} B_j(i\zeta)^j \\ \displaystyle\sum_{j=0}^{2n} C_j(i\zeta)^j & \displaystyle -\sum_{j=0}^{2n+1} A_j(i\zeta)^j \end{pmatrix}\Phi$$

where

$$\text{(10a)} \qquad A_{2n+1} = 4^n, \qquad A_{2k} = 0, \qquad \forall\, k = 0, \dots, n,$$

$$\text{(10b)} \qquad A_{2k+1} = \frac{4^{k+1}}{2}\left\{\mathcal{L}_{n-k}\left[v_x - v^2\right] - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_{n-k-1}\left[v_x - v^2\right]\right\}, \quad k = 0, \dots, n-1,$$

$$\text{(10c)} \qquad B_{2k+1} = \frac{4^{k+1}}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_{n-k-1}\left[v_x - v^2\right], \quad k = 0, \dots, n-1,$$

$$\text{(10d)} \qquad B_{2k} = -4^k\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_{n-k}\left[v_x - v^2\right], \quad k = 0, \dots, n,$$

$$\text{(10e)} \qquad C_{2k+1} = -B_{2k+1}, \quad k = 0, \dots, n-1,$$

$$\text{(10f)} \qquad C_{2k} = B_{2k}, \quad k = 0, \dots, n.$$

Proof. — The compatibility $\Phi_{xt} = \Phi_{tx}$ of equations (9) is guaranteed by the conditions

$$vC - vB = \frac{\partial A}{\partial x}, \tag{11a}$$

$$v_t - 2i\zeta B - 2vA = \frac{\partial B}{\partial x}, \tag{11b}$$

$$v_t + 2i\zeta C + 2vA = \frac{\partial C}{\partial x}. \tag{11c}$$

At the order $\mathcal{O}(1)$ in ζ we obtain

$$v_t = \frac{\partial B_0}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_n\left[v_x - v^2\right],$$

that is (7). We have to show that at each order in ζ^j the compatibility conditions (11) are satisfied. At each order $\mathcal{O}(\zeta^j)$ the conditions (11) give

$$\frac{\partial A_j}{\partial x} = v(C_j - B_j), \tag{12a}$$

$$\frac{\partial B_j}{\partial x} = -2B_{j-1} - 2vA_j, \tag{12b}$$

$$\frac{\partial C_j}{\partial x} = 2C_{j-1} + 2vA_j. \tag{12c}$$

We proceed by induction. At the order $\mathcal{O}(\zeta^{2n+1})$, since by assumption, B_{2n+1} and C_{2n+1} are null, the compatibility conditions give

$$\frac{\partial}{\partial x}A_{2n+1} = 0$$

and by assuming $B_{2n} = -4^n\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_0\left[v_x - v^2\right]$, (12b) gives

$$-4^n\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_0 + vA_{2n+1} = 0.$$

Assuming $A_{2n+1} = 4^n$ the compatibility condition is satisfied because $\mathcal{L}_0 = \frac{1}{2}$. We now assume

$$\begin{aligned} B_{2k+1} &= \frac{4^{k+1}}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_{n-k-1}\left[v_x - v^2\right], \\ C_{2k+1} &= -B_{2k+1}, \end{aligned} \tag{13}$$