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PAINLEVÉ PROPERTY OF THE HÉNON-HEILES HAMILTONIANS

by

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Abstract. — Time independent Hamiltonians of the physical type

$$H = (P_1^2 + P_2^2)/2 + V(Q_1, Q_2)$$

pass the Painlevé test for only seven potentials V, known as the Hénon-Heiles Hamiltonians, each depending on a finite number of free constants. Proving the Painlevé property was not yet achieved for generic values of the free constants. We integrate each missing case by building a birational transformation to some fourth order first degree ordinary differential equation in the classification (Cosgrove, 2000) of such polynomial equations which possess the Painlevé property. The properties common to each Hamiltonian are:

(i) the general solution is meromorphic and expressed with hyperelliptic functions of genus two,

(ii) the Hamiltonian is complete (the addition of any time independent term would ruin the Painlevé property).

***Résumé* (Propriété de Painlevé des hamiltoniens de Hénon-Heiles).** — Les hamiltoniens, indépendants du temps, de la forme

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satisfont au test de Painlevé pour seulement sept potentiels V ; ceux-ci sont connus sous le nom de hamiltoniens de Hénon-Heiles et ils dépendent d'un nombre fini de constantes libres. La propriété de Painlevé restait à établir pour des valeurs génériques des constantes libres. Nous traitons chacun des cas en suspens en construisant une transformation birationnelle vers une équation différentielle ordinaire d'ordre quatre qui figure dans la liste exhaustive (Cosgrove, 2000) de telles équations polynomiales possédant la propriété de Painlevé. Les propriétés communes à ces hamiltoniens sont :

(i) la solution générale est méromorphe et peut être exprimée en termes de fonctions hyperelliptiques de genre deux,

(ii) le hamiltonien est complet au sens où l'addition de tout terme indépendant du temps ferait perdre la propriété de Painlevé.

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1. Introduction

Let us consider the most general two-degree of freedom, classical, time-independent Hamiltonian of the physical type (i.e, the sum of a kinetic energy and a potential energy),

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{1}$$

and let us require that the general solution $q_1^{n_1}, q_2^{n_2}$, with n_1, n_2 integers to be determined, be single valued functions of the complex time t, i.e., what is called the *Painlevé property* of these equations.

A necessary condition is that the Hamilton equations of motion, when written in these variables $q_1^{n_1}, q_2^{n_2}$, pass the Painlevé test ([**12**]). This selects seven potentials V (three "cubic" and four "quartic") depending on a finite number of arbitrary constants, which are known as the Hénon-Heiles Hamiltonians ([**24**]). In order to prove the sufficiency of these conditions, one must then perform the explicit integration and check the singlevaluedness of the general solution. We present here a review on this subject.

The paper is organized as follows:

In section 2, we enumerate the seven cases isolated by the Painlevé test, together with the second constant of the motion K in involution with the Hamiltonian. In section 3, we recall the separating variables in the four cases where they are known. In section 4, we display confluences from quartic cases to all the cubic cases, thus restricting the problem to the consideration of the quartic cases only. In section 5, due to the lack of knowledge of the separating variables in the three remaining cases, we state the equivalence of the equations of motion and the conservation of energy with some fourth order first degree ordinary differential equations (ODEs). In section 6, since these fourth order equations do not belong to any set of already classified equations, we build a birational transformation between each quartic case and some fourth order ODE belonging to a classification of Cosgrove ([**17**]), thus proving the Painlevé property for the quartic cases.

To summarize, the results are twofold:

1. each case is integrated by solving a Jacobi inversion problem involving a hyperelliptic curve of genus two, which proves the meromorphy of the general solution,
2. each case is *complete* in the sense of Painlevé, i.e, it is impossible to add any time-independent term to the Hamiltonian without ruining the Painlevé property.

2. The seven Hénon-Heiles Hamiltonians

By application of the Painlevé test, one isolates two classes of potentials $V(q_1, q_2)$, called "cubic" and "quartic" for simplification.

1. In the cubic case HH3 ([**10, 13, 21**]),

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1 q_1^2 + \omega_2 q_2^2) + \alpha q_1 q_2^2 - \frac{1}{3}\beta q_1^3 + \frac{1}{2}\gamma q_2^{-2}, \quad \alpha \neq 0, \tag{2}$$

in which the constants $\alpha, \beta, \omega_1, \omega_2$ and γ can only take the three sets of values,

$$\text{(SK)}: \quad \beta/\alpha = -1, \omega_1 = \omega_2, \tag{3}$$
$$\text{(KdV5)}: \quad \beta/\alpha = -6, \tag{4}$$
$$\text{(KK)}: \quad \beta/\alpha = -16, \omega_1 = 16\omega_2. \tag{5}$$

2. In the quartic case HH4 ([**23, 32**]),

$$\begin{aligned} H = \frac{1}{2}(P_1^2 + P_2^2 + \Omega_1 Q_1^2 + \Omega_2 Q_2^2) + CQ_1^4 \\ + BQ_1^2 Q_2^2 + AQ_2^4 + \frac{1}{2}\left(\frac{\alpha}{Q_1^2} + \frac{\beta}{Q_2^2}\right) + \gamma Q_1, \quad B \neq 0, \end{aligned} \tag{6}$$

in which the constants $A, B, C, \alpha, \beta, \gamma, \Omega_1$ and Ω_2 can only take the four values (the notation $A : B : C = p : q : r$ stands for $A/p = B/q = C/r =$ arbitrary),

$$\begin{cases} A : B : C = 1 : 2 : 1, & \gamma = 0, \\ A : B : C = 1 : 6 : 1, & \gamma = 0, \ \Omega_1 = \Omega_2, \\ A : B : C = 1 : 6 : 8, & \alpha = 0, \ \Omega_1 = 4\Omega_2, \\ A : B : C = 1 : 12 : 16, & \gamma = 0, \ \Omega_1 = 4\Omega_2. \end{cases} \tag{7}$$

For each of the seven cases so isolated there exists a second constant of the motion K ([**7, 18, 25**]) ([**6, 7, 26**]) in involution with the Hamiltonian,

$$\text{(SK)} \quad K = \left(3p_1 p_2 + \alpha q_2(3q_1^2 + q_2^2) + 3\omega_2 q_1 q_2\right)^2 + 3\gamma(3p_1^2 q_2^{-2} + 4\alpha q_1 + 2\omega_2),$$

$$\begin{aligned} \text{(KdV5)} \quad K = 4\alpha p_2(q_2 p_1 - q_1 p_2) + (4\omega_2 - \omega_1)(p_2^2 + \omega_2 q_2^2 + \gamma q_2^{-2}) \\ + \alpha^2 q_2^2(4q_1^2 + q_2^2) + 4\alpha q_1(\omega_2 q_2^2 - \gamma q_2^{-2}), \end{aligned}$$

$$\begin{aligned} \text{(KK)} \quad K = (3p_2^2 + 3\omega_2 q_2^2 + 3\gamma q_2^{-2})^2 + 12\alpha p_2 q_2^2(3q_1 p_2 - q_2 p_1) \\ - 2\alpha^2 q_2^4(6q_1^2 + q_2^2) + 12\alpha q_1(-\omega_2 q_2^4 + \gamma) - 12\omega_2\gamma, \end{aligned}$$

$$(1:2:1) \quad \begin{cases} K = (Q_2 P_1 - Q_1 P_2)^2 + Q_2^2 \dfrac{\alpha}{Q_1^2} + Q_1^2 \dfrac{\beta}{Q_2^2} \\ \qquad - \dfrac{\Omega_1 - \Omega_2}{2}\left(P_1^2 - P_2^2 + Q_1^4 - Q_2^4 + \Omega_1 Q_1^2 - \Omega_2 Q_2^2 + \dfrac{\alpha}{Q_1^2} - \dfrac{\beta}{Q_2^2}\right), \\ A = \dfrac{1}{2}, \end{cases}$$

$$(1:6:1)\quad \begin{cases} K=\left(P_1P_2+Q_1Q_2\Big(-\dfrac{Q_1^2+Q_2^2}{8}+\Omega_1\Big)\right)^2 \\ \qquad\qquad -P_2^2\dfrac{\kappa_1^2}{Q_1^2}-P_1^2\dfrac{\kappa_2^2}{Q_2^2}+\dfrac{1}{4}\left(\kappa_1^2Q_2^2+\kappa_2^2Q_1^2\right)+\dfrac{\kappa_1^2\kappa_2^2}{Q_1^2Q_2^2}, \\ \alpha=-\kappa_1^2,\quad \beta=-\kappa_2^2,\quad A=-\dfrac{1}{32}, \end{cases}$$

$$(1:6:8)\quad \begin{cases} K=\left(P_2^2-\dfrac{Q_2^2}{16}(2Q_2^2+4Q_1^2+\Omega_2)+\dfrac{\beta}{Q_2^2}\right)^2 \\ \qquad -\dfrac{1}{4}Q_2^2(Q_2P_1-2Q_1P_2)^2+\gamma\Big(-2\gamma Q_2^2-4Q_2P_1P_2 \\ \qquad\qquad +\dfrac{1}{2}Q_1Q_2^4+Q_1^3Q_2^2+4Q_1P_2^2-4\Omega_2Q_1Q_2^2+4Q_1\dfrac{\beta}{Q_2^2}\Big), \\ A=-\dfrac{1}{16}, \end{cases}$$

$$(1{:}12{:}16)\quad \begin{cases} K=\Big(8(Q_2P_1-Q_1P_2)P_2-Q_1Q_2^4-2Q_1^3Q_2^2 \\ \qquad\qquad +2\Omega_1Q_1Q_2^2-8Q_1\dfrac{\beta}{Q_2^2}\Big)^2+\dfrac{32\alpha}{5}\left(Q_2^4+10\dfrac{Q_2^2P_2^2}{Q_1^2}\right), \\ A=-\dfrac{1}{32}., \end{cases}$$

Remark. — *Performing the reduction $q_1=0, p_1=0$ in the three HH3 Hamiltonians (2) yields $H=p^2/2+(1/2)\omega q^2+(1/2)\gamma q^{-2}$, for which q^2 obeys a linearizable Briot-Bouquet ODE. Similarly, the reduction $Q_1=1, P_1=0$ in the four HH4 Hamiltonians (6) yields $H=P^2/2+(1/2)\omega Q^2+AQ^4+(1/2)\beta Q^{-2}$, for which Q^2 obeys the Weierstrass elliptic equation.*

These seven Hénon-Heiles Hamiltonians can be studied from various points of view such as: separation of variables ([**37**]), Painlevé property, algebraic complete integrability ([**3**]). For the interrelations between these various approaches, the reader can refer to the plain introduction in Ref. [**1**]. In the present work, we only deal with proving the Painlevé property (PP).

In order to prove or disprove the PP, it is sufficient to obtain an (explicit) canonical transformation to new canonical variables (the so-called *separating variables*) which separate the *Hamilton-Jacobi equation* for the action $S(q_1,q_2)$ ([**5**, chap. 10]), which for two degrees of freedom is

$$H(q_1,q_2,p_1,p_2)-E=0,\ p_1=\frac{\partial S}{\partial q_1},\ p_2=\frac{\partial S}{\partial q_2}. \tag{8}$$

Indeed, if such separating variables are obtained, depending on the genus g of the hyperellitic curve $r^2 = P(s)$ involved in the associated Jacobi inversion problem,

$$\frac{\mathrm{d}s_1}{\sqrt{P(s_1)}} + \frac{\mathrm{d}s_2}{\sqrt{P(s_2)}} = 0, \qquad \frac{s_1 \mathrm{d}s_1}{\sqrt{P(s_2)}} + \frac{s_2 \mathrm{d}s_2}{\sqrt{P(s_2)}} = \mathrm{d}t, \tag{9}$$

the elementary symmetric functions $s_1 + s_2$ and $s_1 s_2$ are either meromorphic functions of time ($g \leq 2$), or multivalued ($g > 3$).

3. The four cases with known separating variables

Two of the seven cases (KdV5, 1:2:1) have a second invariant K equal to a second degree polynomial in the momenta, therefore there exists a classical method ([**38, 39**]) to obtain the canonical transformation $(q_1, q_2, p_1, p_2) \to (s_1, s_2, r_1, r_2)$ with the separating variables (s_1, s_2) obeying the canonical system (9). For the KdV5 case, one obtains ([**4, 18, 45**])

$$\left\{\begin{array}{ll}
q_1 = -(s_1 + s_2 + \omega_1 - 4\omega_2)/(4\alpha), & q_2^2 = -s_1 s_2/(4\alpha^2), \\
p_1 = -4\alpha \dfrac{s_1 r_1 - s_2 r_2}{s_1 - s_2}, & p_2^2 = -16\alpha^2 \dfrac{s_1 s_2 (r_1 - r_2)^2}{(s_1 - s_2)^2}, \\
H = \dfrac{f(s_1, r_1) - f(s_2, r_2)}{s_1 - s_2}, & \\
f(s, r) = -\dfrac{s^2 (s + \omega_1 - 4\omega_2)^2 (s - 4\omega_2) - 64\alpha^4 \gamma}{32\alpha^2 s} + 8\alpha^2 r^2 s, & \\
f(s_j, r_j) - E s_j + \dfrac{K}{2} = 0,\ j = 1, 2, & \\
P(s) = s^2 (s + \omega_1 - 4\omega_2)^2 (s - 4\omega_2) + 32\alpha^2 E s^2 - 16\alpha^2 K s - 64\alpha^4 \gamma. &
\end{array}\right. \tag{10}$$

For 1:2:1, one obtains

$$\left\{\begin{array}{rcl}
q_j^2 &=& (-1)^j \dfrac{(s_1 + \omega_j)(s_2 + \omega_j)}{\omega_1 - \omega_2}, \quad j = 1, 2, \\
p_j &=& 2 q_j \dfrac{\omega_{3-j}(r_2 - r_1) - s_1 r_1 + s_2 r_2}{s_1 - s_2}, \quad j = 1, 2, \\
H &=& \dfrac{f(s_1, r_1) - f(s_2, r_2)}{s_1 - s_2}, \\
f(s, r) &=& 2(s + \omega_1)(s + \omega_2) r^2 - \dfrac{s^3}{2} - \dfrac{\omega_1 + \omega_2}{2} s^2 \\
&& \quad - \dfrac{\omega_1 \omega_2}{2} s + \dfrac{\omega_2 - \omega_1}{2} \Big(\dfrac{\alpha}{s + \omega_1} - \dfrac{\beta}{s + \omega_2} \Big), \\
f(s_j, r_j) &=& -\Big(s_j + E \dfrac{\omega_1 + \omega_2}{2} \Big) - \dfrac{\alpha + \beta}{2} - \dfrac{K}{2}, \quad j = 1, 2, \\
P(s) &=& s (s + \omega_1)^2 (s + \omega_2)^2 - \alpha (s + \omega_2)^2 - \beta (s + \omega_1)^2 \\
&& \quad -(s + \omega_1)(s + \omega_2) \left[E(2s + \omega_1 + \omega_2) - K \right].
\end{array}\right. \tag{11}$$