Séminaires & Congrès
15, 2007, p. 1–46

BIRKHOFF NORMAL FORM AND HAMILTONIAN PDES

by

Benoît Grébert

Abstract. — These notes are based on lectures held at the Lanzhou University (China) during a CIMPA summer school in July 2004 but benefit from recent developments. Our aim is to explain some normal form technics that allow to study the long time behaviour of the solutions of Hamiltonian perturbations of integrable systems. We are in particular interested with stability results.

Our approach is centered on the Birkhoff normal form theorem that we first proved in finite dimension. Then, after giving some examples of Hamiltonian PDEs, we present an abstract Birkhoff normal form theorem in infinite dimension and discuss the dynamical consequences for Hamiltonian PDEs.

***Résumé* (Forme normale de Birkhoff et EDP hamiltoniennes).** — Ces notes sont basées sur un cours donné à l'université de Lanzhou (Chine) durant le mois de juillet 2004 dans le cadre d'une école d'été organisée par le CIMPA. Cette rédaction bénéficie aussi de développements plus récents. Le but est d'expliquer certaines techniques de forme normale qui permettent d'étudier le comportement pour des temps longs des solutions de perturbations hamiltoniennes de systèmes intégrables. Nous sommes en particulier intéressés par des résultats de stabilité.

Notre approche est centrée sur le théorème de forme normale de Birkhoff que nous rappelons et démontrons d'abord en dimension finie. Ensuite, après avoir donné quelques exemples d'EDP hamiltoniennes, nous démontrons un théorème de forme normale de Birkhoff en dimension infinie et nous en discutons les applications à la dynamique des EDP hamiltoniennes.

1. Introduction

The class of Hamiltonian systems close to integrable system contain most of the important physic models. Typically a Hamiltonian system in finite dimension reads (cf. section 2)

$$\begin{cases} \dot q_j = \frac{\partial H}{\partial p_j}, & j = 1, \dots, n \\ \dot p_j = -\frac{\partial H}{\partial q_j}, & j = 1, \dots, n \end{cases}$$

2000 ***Mathematics Subject Classification.*** — 37K55, 37K45, 35B34, 35B35.
Key words and phrases. — Birkhoff normal form, Resonances, KAM theory, Hamiltonian PDEs, long time stability.

where the Hamiltonian H is a smooth fonction from $\mathbb{R}^{2N}$ to $\mathbb{R}$. In these lectures we are interesred in the case where H decomposes in $H = H_0 + \epsilon P$, H_0 being integrable in the sense that we can "integrate" the Hamiltonian system associated to H_0 (cf. section 2.3), P being the perturbation and ϵ a small parameter. This framework contains a lot of important example of the classical mechanics. If we allow the number of degrees of freedom, N, to grow to infinity, then we arrive in the world of quantum mechanics and the corresponding equations are typically nonlinear partial differential equations (PDEs). Again a lot of classical examples are included in this framework like, for instance, the nonlinear wave equation, the nonlinear Schrödinger equation or the Korteweg-de Vries equation (cf. section 5.1).

The historical example (in finite dimension) is given by the celestial mechanics: More than 300 years ago Newton gaves the evolution equation for a system of N heavy bodies under the action of the gravity.

When $N = 2$, Kepler gaves the solution, the bodies describe ellipses. Actually for $N = 2$ the system is integrable.

As soon as $N \geq 3$ the system leaves the integrable world and we do not know the expression of the general solution. Nevertheless if we consider the celestial system composed by the Sun (S), the Earth (E) and Jupiter (J) and if we neglect the interaction between J and E, then the system is again integrable and we find quasiperiodic solution. Mathematically the solutions read $t \mapsto g(\omega_1 t, \omega_2 t, \omega_3 t)$ where g is a regular function from the torus $T^3 = S^1 \times S^1 \times S^1$ to $\mathbb{R}^{18}$ (three postions and three moments in $\mathbb{R}^3$) and ω_j, $j = 1, 2, 3$ are frequencies. Visually J and E turn around S which turns around the center of mass. Notice that the trajectory (or orbit) is contained in the torus $g(\mathcal{T}^3)$ of dimension 3 and that this torus is invariant under the flow. On the other hand, if $(\omega_1, \omega_2, \omega_3)$ are rationnaly independent, then the trajectory densely fills this torus while, if for instance the three frequencies are rationnally proportional, then the trajectory is periodic and describes a circle included in $g(\mathcal{T}^3)$.

Now the exact system S-E-J is described by a Hamiltonian $H = H_0 + \epsilon P$ in which H_0 is the integrable Hamiltonian where we neglect the interaction E-J, P takes into account this interaction and $\epsilon = \frac{\text{jupiter's mass + earth's mass}}{\text{sun's mass}}$ plays the rule of the small parameter.

Some natural questions arrive:

- Do invariant tori persist after this small perturbation?
- At least are we able to insure stability in the sense that the planets remain in a bounded domain?
- Even if we are unable to answer these questions for eternity, can we do it for very large – but finite – times?

These questions have interested a lot of famous mathematicians and physicists. In the 19-th century one tried to expand the solutions in perturbative series: $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$, the term u_{k+1} being determined by an equation involving $u_0, \ldots, u_k$. Unfortunatly this series does not converge. This convergence problems seemed so involved that, at the principle of the 20-th century, most of scientist believed in the *ergodic hypothesis*: typically, after arbitrarily small perturbation, all the trajectories fill all the phase space and the stable trajectories are exceptionnal. Actually, H. Poinaré proved that a dense set of invariant tori are destroyed by an arbitrarily small perturbation. Nevertheless, a set can be dense but very small and in 1954 A. N. Kolmogorov [**Kol54**] announced that the majority (in the measure sense) of tori survive (see section 7). The proof of this result was completed by V. Arnold [**Arn63**] and J. Moser [**Mos62**] giving birth to the KAM theory.

In order to illustrate this result we can apply it to a simplified S-E-J system: we assume that the S-E-J system reduces to a Hamiltonian system with 3 degrees of freedom without symmetries (the symmetries of the true system complicates the pictures and generates degenerancies). In this case, the KAM theorem says, roughly speaking (see theorem 7.4 for a precise statement), that if $(\omega_1, \omega_2, \omega_3) \in \mathcal{C}$, a Cantor set of $\mathbb{R}^3$ having a positive measure, or equivalently if the initial positions and moments are in a Cantor set, then the trajectory is quasi periodic. Since a Cantor set has an empty interior the condition $(\omega_1, \omega_2, \omega_3) \in \mathcal{C}$ is not physical (no measurement could decide if this condition is verified or not).

The present lectures will be centered on the Birkhoff normal form approach which does not control the solution for any times but does not require an undecidable hypothesis. In the case of our simplified S-E-J system, the Birkhoff normal form theorem says, roughly speaking, that having fixed an integer $M \geq 1$, and $\epsilon < \epsilon_0(M)$ small enough, to any initial datum corresponding to not rationaly dependent frequencies $(\omega_1, \omega_2, \omega_3)$, we can associate a torus such that the solution remains ϵ-close to that torus during a lapse of time greater than $1/\epsilon^M$ (see section 3 for a precise statement). Note that this result can be physically sufficient if $1/\epsilon^M$ is greater than the age of the universe.

The rational independence of the frequencies (one also says the nonresonancy) is of course essential in all this kind of perturbative theorems. Again we can illustrate this fact with our system S-E-J: suppose that, when considering the system without E-J interaction, the three bodies are periodically align, the Earth being between Jupiter and the Sun (notice that this implies that the frequencies $(\omega_1, \omega_2, \omega_3)$ are rationaly dependent). When we turn on the interaction E-J, Jupiter will attract the Earth outside of its orbit periodically (i.e., when the three bodies are align or almost align), these accumulate small effects will force the earth to escape its orbit and thus the invariant torus will be destroyed.

The generalisation of these results to the infinite dimensional case is of course not easy but it worth trying: The expected results may apply to nonlinear PDEs when they can be viewed as an infinite dimensional Hamiltonian system (cf. section 5) and concern the long time behaviour of the solution, a very difficult and competitive domain.

For a general overview on Hamiltonian PDEs, the reader may consult the recent monographies by Craig [**Cra00**], by Kuksin [**Kuk00**], by Bourgain [**Bou05a**] and by Kappleler and Pöschel [**KP03**]. In the present lectures we mainly focus on the extension of the Birkhoff normal form theorem. Such extension was first (partially) achieved by Bourgain [**Bou96**] and then by Bambusi [**Bam03**]. The results stated in this text was first proved by Bambusi and myself in [**BG06**]. The proof presented here and some generalisations benefit of a recent collaboration with Delort and Szeftel [**BDGS07**].

After this general presentation, I give a brief outline of the next sections:

Section 2 : We recall briefly the classical Hamiltonian formalism including: integrals of the motion, Lie transforations, Integrability in the Liouville sense, action angle variables, Arnold-Liouville theorem (see for instance [**Arn89**] for a complete presentation).

Section 3 : We state and prove the Birkhoff normal form theorem and then present its dynamical consequences. Theses results are well known and the reader may consult [**MS71, HZ94, KP03**] for more details and generalizations.

Section 4 : We state a Birkhoff normal form theorem in infinite dimension and explain its dynamical consequences. In particular, results on the longtime behaviour of the solutions are discussed. This is the most important part of this course. A slightly more general abstract Birkhoff theorem in infinite dimension was obtained in [**BG06**] and the dynamical consequences was also obtained there.

Section 5 : Two examples of Hamiltonian PDEs are given: the nonlinear wave equation and the nonlinear Schrödinger equation. We then verify that our Birkhoff theorem and its dynamical consequences apply to both examples.

Section 6 : Instead of giving the proof of [**BG06**], we present a simpler proof using a class of polynomials first introduced in [**DS04**], [**DS06**]. Actually we freely used notations and parts of proofs of theses three references.

Section 7 : In a first part we comment on some generalisations of our result. In the second subsection, we try to give to the reader an idea on the KAM theory in both finite and infinite dimension. Then we compare the Birkhoff approach with the KAM approach.

Acknowledgements. — It is a great pleasure to thank D. Bambusi and J. M. Delort for helpful discussions on these notes.

2. Hamiltonian formalism in finite dimension

2.1. Basic definitions. — We only consider the case where the **phase space** (or configuration space) is an open set, M, of $\mathbb{R}^{2n}$. We denote by J the canonical

Poisson matrix, i.e.,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} .$$

More generally, J could be an antisymmetric matrix on $\mathbb{R}^{2n}$. All the theory can be extended to the case where the phase space is a $2n$ dimensional symplectic manifold.

A **Hamiltonian fonction**, H, is a regular real valued function on the phase space, i.e., $H \in C^\infty(M, \mathbb{R})$. To H we associate the **Hamiltonian vector field**

$$X_H(q,p) = J\nabla_{q,p}H(q,p)$$

where $\nabla_{p,q}H$ denotes the gradient of H with respect to p, q, i.e.,

$$\nabla_{q,p}H = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix} , \quad X_H = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial q_1} \\ \vdots \\ -\frac{\partial H}{\partial q_n} \end{pmatrix} .$$

The associated **Hamiltonian system** then reads

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = X_H(q,p)$$

or equivalently

$$\begin{cases} \dot q_j = \frac{\partial H}{\partial p_j}, & j = 1, \dots, n\,, \\ \dot p_j = -\frac{\partial H}{\partial q_j}, & j = 1, \dots, n\,. \end{cases}$$

The **Poisson bracket** of two Hamiltonian functions F, G is a new Hamiltonian function $\{F, G\}$ given by

$$\{F,G\}(q,p) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(q,p)\frac{\partial G}{\partial p_j}(q,p) - \frac{\partial F}{\partial p_j}(q,p)\frac{\partial G}{\partial q_j}(q,p) \;.$$

2.2. A fundamental example: the harmonic oscillator. — Let $M = \mathbb{R}^{2n}$ and

$$H(q,p) = \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2}$$

where

$$\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} \in \mathbb{R}^n$$

is the frequencies vector. The associated system is the **harmonic oscillator** whose equations read

$$\begin{cases} \dot q_j = \omega_j p_j, & j = 1, \dots, n \\ \dot p_j = -\omega_j q_j, & j = 1, \dots, n \end{cases}$$