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LARGE TIME BEHAVIOR IN PERFECT INCOMPRESSIBLE FLOWS

by

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Abstract. — We present in these lecture notes a few recent results about the large time behavior of solutions of the Euler equations in the full plane or in a half plane. We will investigate the confinement properties of the vorticity and we will try to determine the structure of the weak limit of different rescalings of the vorticity.

***Résumé* (Comportement en temps grand pour les fluides parfaits incompressibles)**
Nous présentons dans ces notes de cours quelques résultats récents sur le comportement en temps grand des solutions des équations d'Euler dans le plan entier ou dans un demi-plan. Nous étudions les propriétés de confinement du tourbillon et nous essaierons de déterminer la structure de la limite faible de divers changements d'échelle du tourbillon.

1. Introduction

These lecture notes correspond to an eight hours mini-course that the author taught at the CIMPA summer school in Lanzhou (China) during July 2004.

The equation of motion of a perfect incompressible fluid were deduced by Euler [**13**] by assuming that there is no friction between the molecules of the fluid. In the modern theory of existence and uniqueness of solutions, the case of the dimension two is by far the richest one. Global existence and uniqueness of bidimensional solutions was first proved by Wolibner [**42**] for smooth initial data and by Yudovich [**45**] for data with bounded vorticity. There are also some global existence results (no uniqueness yet) when the vorticity belongs to L^p or is a nonnegative compactly supported H^{-1} Radon measure. As far as the dimension three is concerned, only some local in time results are known, except in some very particular cases.

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After obtaining this global existence theory in dimension two under more or less satisfactory hypothesis, a natural question arises: what is the large time behavior of these solutions? Unfortunately, the answer to this question is still largely unknown. The few results that are known give some information on the vorticity rather than the velocity itself. This 8 hours mini-course is intended to present the latest developments on the subject together with a introduction to the equations and a review of the main global existence of solutions results.

The structure of these notes is the following. In Part I we start by giving a very short presentation of the equations, we introduce the main quantities and list without proof the conservations laws that will be used in the sequel. Next we review the most important global existence and uniqueness of solutions results; the main ideas of the proofs are also highlighted. After this introductory part, we discuss in Part II some relevant examples of solutions for the Euler equations and the vortex model; the behavior observed here will be precious in the sequel. Part III deals with the confinement properties of nonnegative vorticity. We end this work with the most general case, the case of unsigned vorticity. Here we will find another point of view for the large time behavior: we will try to describe the weak limits of different rescalings of the vorticity.

Part I is given only to make these lecture notes self-contained. For these reasons, the write-up is rather sketchy with very few details given. The main part of this work consists of Parts II, III and IV which are more complete and carefully written.

PART I
PRESENTATION OF THE EQUATIONS AND EXISTENCE OF SOLUTIONS

2. Presentation of the equations, Biot-Savart law and conserved quantities

Let u be the velocity of a perfect incompressible fluid filling $\mathbb{R}^n$ and p the pressure. Assuming that the density is constant equal to 1, the vector field u and the scalar function p must satisfy the following Euler equation

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0, \quad u\big|_{t=0} = u_0,$$

where $\operatorname{div} u = \sum_i \partial_i u_i$ and $u \cdot \nabla = \sum_i u_i \partial_i$. If we place ourselves on a bounded domain, then we must also assume the so-called slip boundary conditions which say that the velocity is tangent to the boundary and express the fact that the boundary is not permeable. We define the vorticity to be the following antisymmetric matrix

$$\Omega = (\partial_j u_i - \partial_i u_j)_{i,j}.$$

In dimension 2 we identify Ω to a scalar function,

$$\Omega \equiv \omega = \partial_1 u_2 - \partial_2 u_1$$

while in dimension 3 we identify it with the following vector field.

$$\Omega \equiv \omega = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}.$$

From the divergence free condition on u, one can check that

$$\triangle u = \operatorname{div} \Omega = \Big(\sum_j \partial_j \Omega_{ij} \Big)_i$$

Using the formula for the fundamental solution of the Laplacian in $\mathbb{R}^n$ we deduce the following formula expressing the velocity in terms of the vorticity.

$$u = C_n \int_{\mathbb{R}^n} \Omega(y) \frac{x-y}{|x-y|^n} \,\mathrm{d}y.$$

The above relation is called the Biot-Savart law. In dimension 2, the Biot-Savart law can be expressed as follows:

$$u = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi |x-y|^2} \omega(y) \,\mathrm{d}y = \frac{x^\perp}{2\pi |x|^2} * \omega,$$

where $x^\perp = (-x_2, x_1)$.

It is a simple calculation to check that the vorticity equation is

$$\partial_t \Omega + u \cdot \nabla \Omega + (\nabla u)\Omega + \Omega (\nabla u)^t = 0$$

while in dimension 2 it can be expressed as a simple transport equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{1}$$

From this transport equation it is not difficult to deduce that the following quantities are conserved in dimension 2:

- $\int_{\mathbb{R}^2} u$;
- the energy $\|u\|_{L^2}^2$ and the generalized energy $\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \log|x-y| \omega(x)\omega(y) \,\mathrm{d}x \,\mathrm{d}y$;
- $\int_{\mathbb{R}^2} \omega$ and all L^p norms of ω, $1 \le p \le \infty$;
- center of mass $\int_{\mathbb{R}^2} x\omega(x) \,\mathrm{d}x$;
- moment of inertia $\int_{\mathbb{R}^2} |x|^2 \omega(x) \,\mathrm{d}x$;
- circulation on a material curve $\int_\Gamma u \cdot \mathrm{d}s$ (Γ is a curve transported by the flow).

3. Existence and uniqueness results

The aim of this section is to give a review of the most important global existence (and sometimes uniqueness) of bidimensional solutions to the Euler equations and also to give a very short sketch of the proof with the main ingredients. We start with the case of classical solutions in Subsection 3.1, we continue with L^p vorticities in Subsection 3.2 and we end with the very interesting case of vortex sheets in Subsection 3.3.

3.1. Strong solutions and the blow-up criterion of Beale-Kato-Majda. — We first deal with strong solutions that belong to the Sobolev space $H^m(\mathbb{R}^n)$, $m > \frac{n}{2}+1$. By Sobolev embeddings, such a solution is C^1 so it verifies the equation in the classical sense. Their existence is in general only local in time, but the Beale, Kato and Majda [**3**] blow-up criterion ensures that the existence is global in dimension 2. More precisely, we have the following result.

Theorem 3.1. — *Suppose that the initial velocity u_0 is divergence free and belongs to the Sobolev space $H^m(\mathbb{R}^n)$ where $m > \frac{n}{2}+1$. There exists a unique local solution $u \in C^0\big([0,T);H^m\big)$ with $T \geq \frac{C}{\|u_0\|_{H^m}}$. Moreover, the following blow-up criterion due to Beale, Kato and Majda holds: if T^*, the maximal time existence of this local solution, is finite, then $\int_0^{T^*} \|\Omega\|_{L^\infty} = \infty$.*

Corollary 3.2. — *In dimension 2 the above solution is global.*

Proof of the corollary. — The proof is trivial from the Beale, Kato and Majda blow-up criterion since the L^∞ norm of the vorticity is conserved. □

Sketch of proof of Theorem 3.1. — The *a priori* estimates

$$\partial_t \|u\|_{H^m}^2 \leq C\|u\|_{H^m}^2 \|\nabla u\|_{L^\infty}$$

follow from the following Gagliardo-Nirenberg inequality

$$\|D^\ell u\|_{L^{\frac{2k}{\ell}}} \leq C\|u\|_{L^\infty}^{1-\frac{\ell}{k}} \|D^k u\|_{L^2}^{\frac{\ell}{k}}, \qquad 0 \leq \ell \leq k,$$

and from the cancellation $\int u \cdot \nabla D^m u D^m u = 0$. The first part of the theorem follows from the Sobolev embedding $H^{m-1} \subset L^\infty$ used to estimate $\|\nabla u\|_{L^\infty} \leq C\|u\|_{H^m}$.

We now prove the blow-up condition. Assume, by absurd, that $\int_0^{T^*} \|\Omega\|_{L^\infty} < \infty$. From the vorticity equation and using that $\|\nabla u\|_{L^2} \simeq \|\Omega\|_{L^2}$, one can easily deduce that $\Omega \in L^\infty(0,T^*;L^2)$. We now use the following standard logarithmic inequality

$$\|\nabla u\|_{L^\infty} \leq C[1 + \|\Omega\|_{L^2} + \|\Omega\|_{L^\infty}(1 + \log_+ \|u\|_{H^m})]$$

to deduce that

$$\|\nabla u\|_{L^\infty} \leq C(1 + \|\Omega\|_{L^\infty} \int_0^t \|\nabla u\|_{L^\infty}).$$

Gronwall's inequality therefore implies that $\int_0^{T^*} \|\nabla u\|_{L^\infty} < \infty$ which in turn gives that $u \in L^\infty(0, T^*; H^m)$ which obviously contradicts the maximality of T^*. ☐

3.2. Solutions with compactly supported L^p vorticity. — From now on we assume that the space dimension is equal to two. Let L^p_c denote the space of compactly supported L^p functions. If $p > 1$ and $\omega_0 \in L^p_c$ then $\omega \in L^\infty(\mathbb{R}_+; L^p)$ and therefore $u \in L^\infty(\mathbb{R}_+; W^{1,p}_{loc})$. Global existence of solutions follows with a standard approximation procedure and basically from the compact embedding $W^{1,p}_{loc} \hookrightarrow L^2_{loc}$, see [**12**]. Uniqueness of these solutions is not known unless $p = \infty$ when the following uniqueness result due to Yudovich [**45**] holds.

***Theorem 3.3* (Yudovich).** — *Suppose that $\omega_0 \in L^\infty_c$. There exists a unique global solution such that $\omega \in L^\infty(\mathbb{R}_+; L^\infty_c)$.*

Sketch of proof of uniqueness. — The proof relies on the following well-known singular integral estimate:

$$\|\nabla u\|_{L^p} \leq Cp\|\omega\|_{L^p} \qquad \forall 2 \leq p < \infty.$$

Let u and v be two solutions and set $w = u - v$. Then

$$\partial_t w + u \cdot \nabla w + w \cdot \nabla v = \nabla p'.$$

We now make L^2 energy estimates on this equation by multiplying with w to obtain

$$\partial_t \|w\|^2_{L^2} = -2 \int w \cdot \nabla v w \leq 2\|w\|_{L^2} \|\nabla v\|_{L^p} \|w\|_{L^{\frac{2p}{p-2}}} \leq Cp\|w\|^{2-\frac{2}{p}}_{L^2}.$$

After integration we get $\|w(t)\|_{L^2} \leq (Ct)^p$. Sending $p \to \infty$ yields $w\big|_{[0,\frac{1}{C}]} = 0$. Global uniqueness follows by repeating this argument. ☐

3.3. Vortex sheets and the Delort theorem. — The vortex sheet problem appears when the velocity has a jump over an interface. In this case, the vorticity is no longer a function but a measure since it must contain the Dirac mass of the interface. Previous global existence results do not apply. Nevertheless, we have the following very important global existence result due to Delort [**11**].

***Theorem 3.4* (Delort).** — *Suppose that $u_0 \in L^2_{loc}(\mathbb{R}^2)$ is such that the initial vorticity ω_0 is a nonnegative compactly supported Radon measure. Then there exists a global solution $u \in L^\infty_{loc}(\mathbb{R}_+; L^2_{loc})$.*

Sketch of proof. — We give here the main ideas of the version of the proof given by Schochet [**40**]. First of all, it is very easy to see by standard energy estimates that *a priori* $u \in L^\infty_{loc}(\mathbb{R}_+; L^2_{loc})$ which implies that $\omega \in L^\infty_{loc}(\mathbb{R}_+; H^{-1}_{loc})$.