Séminaires & Congrès
15, 2007, p. 309–331

SOME LIMITING SITUATIONS FOR SEMILINEAR ELLIPTIC EQUATIONS

by

Dong Ye

Abstract. — The objective of this mini-course is to take a look at a standard semilinear partial differential equation $-\Delta u = \lambda f(u)$ on which we show the use of some basic tools in the study of elliptic equation. We will mention the maximum principle, barrier method, blow-up analysis, regularity and boot-strap argument, stability, localization and quantification of singularities, Pohozaev identities, moving plane method, etc.

***Résumé* (Quelques situations limites pour les équations semi-linéaires elliptiques)**
L'objectif de ce mini-cours est de jeter un coup d'œil sur une équation aux dérivées partielles standard $-\Delta u = \lambda f(u)$, avec laquelle nous allons montrer quelques outils de base dans l'étude des équations elliptiques. Nous mentionnerons le principe du maximum, la méthode de barrière, l'analyse de blow-up, la régularité, l'argument de boot-strap, la stabilité, la localisation et quantification de singularités, les identités de Pohozaev, la méthode du plan mobile, etc.

1. Introduction

We consider the following semilinear partial differential equation:

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and f is a smooth positive, nondecreasing and convex function over $\mathbb{R}_+$. For getting a positive solution u, necessarily λ is positive.

The convexity of f implies that

- $\lim_{t\to\infty} f(t)/t = a \in \mathbb{R}_+ \cup \{\infty\}$ exists.
- If $a \in \mathbb{R}_+$, then $\lim_{t\to\infty} f(t) - at = l \in \mathbb{R} \cup \{-\infty\}$ exists.

2000 ***Mathematics Subject Classification.*** — 35J60, 35B40.
Key words and phrases. — Semilinear elliptic PDE, blow-up analysis, regularity and singularity.

Since the case $a = 0$ is trivial ($f \equiv$ *constant*), we will suppose that $a > 0$. Thus we can divide the study of problem (P_λ) into two different situations: the quasilinear case when $a \in (0, \infty)$ and superlinear case when $a = \infty$. We will see that the first case is rather well understood, while many questions are remained open for the second one.

In the following, $\|\cdot\|_p$ denotes the standard L^p norm for $1 \leq p \leq \infty$. $W^{1,p}(\Omega)$ is the Sobolev space of functions f such that f and $\nabla f \in L^p(\Omega)$. When $p = 2$, we use for simplicity $H^1(\Omega)$ to denote $W^{1,2}(\Omega)$, $H_0^1(\Omega)$ is the space of functions $f \in H^1(\Omega)$ verifying $f = 0$ on $\partial\Omega$. The symbol C means always a positive constant independent of λ.

2. Quasilinear situation

We begin with the quasilinear case where $a \in (0, \infty)$. Many results presented here are obtained by Mironescu & Rădulescu in [**27**].

2.1. Minimal solution and stability. — Since $f(u) \leq au + f(0)$ in this case, then if $u \in L^1(\Omega)$ is a weak solution of (P_λ) in the sense of distribution, we get easily that u is always a classical solution by standard boot-strap argument.

Lemma 2.1. — *For $\lambda > 0$, if (P_λ) is resolvable, then a minimal solution u_λ exists in the sense that any solution v of (P_λ) verifies $v \geq u_\lambda$ in Ω. Moreover, $(P_{\lambda'})$ is resolvable for any $\lambda' \in (0, \lambda)$.*

Proof. — We will use the barrier method. Remark that for $\lambda > 0$, $w_0 \equiv 0$ is a sub solution of (P_λ) since $f(0) > 0$. Now we define for any $n \in \mathbb{N}$,

$$-\Delta w_{n+1} = \lambda f(w_n) \ \text{ in } \Omega, \quad w_{n+1} = 0 \ \text{ on } \partial\Omega. \tag{1}$$

Using maximum principle, $w_1 > w_0 \equiv 0$ in Ω. On the other hand, let v be any solution of (P_λ), by monotonicity of f, we obtain

$$-\Delta(w_1 - v) = \lambda\,[f(0) - f(v)] \leq 0 \ \text{ in } \Omega, \quad w_1 - v = 0 \ \text{ on } \partial\Omega.$$

Thus $w_1 \leq v$ in Ω. We can prove by induction that the sequence $\{w_n\}$ verifies $w_n \leq w_{n+1} \leq v$ in Ω for any n, so $u_\lambda = \lim_{n\to\infty} w_n$ is well defined, and u_λ is a solution of (P_λ) by passing to the limit in (1). Moreover, $u_\lambda \leq v$. Notice that the definition of u_λ is independent of the choice of v, it is the minimal solution claimed.

If (P_λ) has a solution u, it is a super solution for $(P_{\lambda'})$ when $0 < \lambda' < \lambda$. As $\omega_0 \equiv 0$ is always a sub solution, the barrier method will solve as above $(P_{\lambda'})$. □

Let λ_1 be the first eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition, we define φ_0 to be the first eigenfunction such that $\varphi_0 > 0$ in Ω and $\|\varphi_0\|_2 = 1$.

Lemma 2.2. — *If we denote $r_0 = \inf_{t>0} f(t)/t$, then (P_λ) has no solution for $\lambda > \lambda_1/r_0$. On the other hand, (P_λ) is resolvable for $\lambda > 0$ small enough.*

Proof. — Let $\xi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the solution of $-\Delta\xi = 1$ in Ω. It is easy to see that ξ is a super solution of (P_λ) for $\lambda \leq f(\|\xi\|_\infty)^{-1}$. Applying the barrier method, we get a solution of (P_λ) for such λ.

Now we suppose that u is a solution of (P_λ) for some $\lambda > 0$, using φ_0 as test function and integrating by parts, we get

$$\lambda_1 \int_\Omega \varphi_0 u dx = -\int_\Omega u\Delta\varphi_0 dx = -\int_\Omega \varphi_0 \Delta u dx = \lambda \int_\Omega f(u)\varphi_0 dx.$$

As $f(u) \geq r_0 u$ in Ω, we have then

$$(\lambda_1 - \lambda r_0)\int_\Omega \varphi_0 u dx \geq 0.$$

Recalling that φ_0 and u are positive in Ω, the lemma is proved. □

Combining these two lemmas, we can claim

Theorem 2.3. — *There exists a critical value $\lambda^* \in (0, \infty)$ for the parameter λ, such that for any $\lambda > \lambda^*$, no solution exists for the problem (P_λ) while for any $\lambda \in (0, \lambda^*)$, a unique minimal solution u_λ exists for (P_λ). Furthermore the mapping $\lambda \mapsto u_\lambda$ is increasing with λ.*

It is natural to ask if we can determine the exact value of λ^* and what happens when $\lambda = \lambda^*$. Before considering these two questions, we show another characterization of the minimal solution u_λ, its stability. A solution u of (P_λ) is called stable if and only if the linearized operator associated to the equation, $-\Delta - \lambda f'(u)$ is nonnegative. More precisely,

$$\lambda \int_\Omega f'(u)\varphi^2 dx \leq \int_\Omega |\nabla\varphi|^2 dx, \quad \text{for any } \ \varphi \in H_0^1(\Omega). \tag{2}$$

Theorem 2.4. — *Let $\lambda \in (0, \lambda^*)$, the minimal solution u_λ is the unique stable solution of (P_λ).*

Proof. — First we prove that u_λ is stable. If it is not true, the first eigenvalue η_1 of $-\Delta - \lambda f'(u_\lambda)$ is negative, then there exists an eigenfunction $\psi \in H_0^1(\Omega)$ such that

$$-\Delta\psi - \lambda f'(u_\lambda)\psi = \eta_1\psi \ \text{ in } \Omega, \quad \psi > 0 \ \text{ in } \Omega.$$

Consider $u^\varepsilon = u_\lambda - \varepsilon\psi$, a direct calculation gives

$$-\Delta u^\varepsilon - \lambda f(u^\varepsilon) = -\eta_1\varepsilon\psi - \lambda\left[f(u_\lambda - \varepsilon\psi) - f(u_\lambda) + \varepsilon f'(u_\lambda)\psi\right] = \varepsilon\psi\left[-\eta_1 + o_\varepsilon(1)\right].$$

Since $\eta_1 < 0$, then $-\Delta u^\varepsilon - \lambda f(u^\varepsilon) \geq 0$ in Ω for $\varepsilon > 0$ small enough. Otherwise, using Hopf's lemma, we know that $u_\lambda \geq C\psi$ in Ω for some $C > 0$. Thus $u^\varepsilon \geq 0$ is a super solution of (P_λ) for $\varepsilon > 0$ small enough. As before, we can get a solution u such that $u \leq u^\varepsilon$ in Ω, which contradicts the minimality of u_λ. So $\eta_1 \geq 0$.

Now we prove that (P_λ) has at most one stable solution. Suppose the contrary, there exists another stable solution $v \neq u_\lambda$. Define $\varphi = v - u_\lambda$, we get

$$\lambda\int_\Omega f'(v)\varphi^2 dx \leq \int_\Omega |\nabla\varphi|^2 dx = -\int_\Omega \varphi\Delta\varphi dx = \lambda\int_\Omega \left[f(v) - f(u_\lambda)\right]\varphi dx,$$

so
$$\int_\Omega [f(v) - f(u_\lambda) - f'(v)(v-u_\lambda)]\,\varphi dx \geq 0.$$
By maximum principle, we know that $\varphi > 0$ in Ω. The convexity of f yields that the term in the bracket is non positive, so the only possibility is $f(v) - f(u_\lambda) - f'(v)(v-u_\lambda) \equiv 0$ in Ω, which means f is affine over $[u_\lambda(x), v(x)]$ for any $x \in \Omega$. Thus $f(x) = \bar{a}x + b$ in $[0, \max_\Omega v]$ and we get two solutions u and v of $-\Delta w = \bar{a}w + b$. This implies that
$$0 = \int_\Omega u_\lambda \Delta v - v\Delta u_\lambda dx = b\int_\Omega (v-u_\lambda)dx = b\int_\Omega \varphi dx,$$
which is impossible since $b = f(0) > 0$ and φ is positive in Ω. So we are done. □

An immediate consequence of Theorem 2.4 is

Proposition 2.5. — *For any $\lambda \in (0, \lambda_1/a)$, (P_λ) has one and unique solution u_λ.*

Proof. — Remark first $a = \sup_{\mathbb{R}_+} f'(t)$ by convexity of f. Thanks to the definition of λ_1, it is clear that each solution is stable if $\lambda \in (0, \lambda_1/a)$, so we get the uniqueness by that for stable solution. For the existence, we can consider the minimization problem $\min_{H_0^1(\Omega)} J(u)$ where
$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda\int_\Omega F(u)dx$$
with
$$F(u) = \int_0^{u^+} f(s)ds, \quad u^+ = \max(u, 0).$$
If $\lambda \in (0, \lambda_1/a)$, there exist ε, $A > 0$ depending on λ such that $2\lambda F(t) \leq (\lambda_1 - \varepsilon)t^2 + A$ over $\mathbb{R}$. Thus $J(u)$ is coercive, bounded from below and weakly lower semi-continuous in $H_0^1(\Omega)$, the infimum of J is reached then by a function $u \in H_0^1(\Omega)$, so also by $u^+ \in H_0^1(\Omega)$ since $J(u^+) \leq J(u)$. This critical point $u \geq 0$ of J gives a solution of (P_λ). □

2.2. Estimate of λ^*. — By Proposition 2.5, we know that $\lambda^* \geq \lambda_1/a$. The following result in [**27**] gives us more precise information for λ^*.

Theorem 2.6. — *We have three equivalent assertions:*

(i) $\lambda^* = \lambda_1/a$.
(ii) *No solution exists for (P_{λ^*}).*
(iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *u.c. in Ω. (u.c. means "uniformly on each compact set")*

Proof. — *(i) implies (ii)*. If (P_{λ^*}) has a solution u, then $u_\lambda \leq u$ in Ω for any $\lambda \in (0, \lambda^*)$, using the monotonicity of u_λ, $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is well defined and u^* is clearly a stable solution of (P_{λ^*}) by limit. Consider the operator $G(u, \lambda) = -\Delta u - \lambda f(u)$, if the first eigenvalue η_1 of $-\Delta - \lambda^* f'(u^*)$ is positive, then we can apply the Implicit Function Theorem to get a solution curve in a neighborhood of λ^*, but this contradicts the definition of λ^*, so $\eta_1 = 0$. Thus, there exists $\psi \in H_0^1(\Omega)$ satisfying
$$-\Delta\psi - \lambda^* f'(u^*)\psi = 0 \quad \text{and} \quad \psi > 0 \quad \text{in } \Omega. \tag{3}$$

Using φ_0 as test function and integrating by parts, we get

$$\int_\Omega [\lambda_1 - \lambda^* f'(u^*)]\, \psi\varphi_0 dx = 0.$$

As $\lambda_1 - \lambda^* f'(u^*) \geq 0$, we get $f'(u^*) \equiv a$ in Ω so that $f(t) = at + b$ in $[0, \max_\Omega u^*]$. But $b > 0$ deduces that no positive solution in $H_0^1(\Omega)$ can exist for the equation $-\Delta u = \lambda_1 u + b\lambda_1/a$ (we can use again φ_0), so the hypothesis is not true.

(ii) implies (iii). Here we mention a result of Hörmander (see [**22**]) as follows: *For a sequence of nonnegative super-harmonic functions $\{v_n\}$ in Ω, either v_n converges u.c. to ∞; or there exists a subsequence which converges in $L^1_{loc}(\Omega)$.* We need just to prove that the second case cannot occur for u_λ. Suppose the contrary, there exist $u_k = u_{\lambda_k}$ which converges in $L^1_{loc}(\Omega)$ to u^* with $\lambda_k \to \lambda^*$. We claim that $\|u_k\|_2 \leq C$. If it is false, we define $u_k = l_k w_k$ with $\|w_k\|_2 = 1$ and $\lim_{k\to\infty} l_k = \infty$ (up to subsequence). Since $f(t) \leq at + f(0)$,

$$-\Delta w_k = \frac{\lambda_k f(u_k)}{l_k} \leq a\lambda_k w_k + \frac{\lambda_k f(0)}{l_k} \leq a\lambda_k w_k + C \quad \text{in } \Omega,$$

it is easy to see that w_k is bounded in $H_0^1(\Omega)$, so that up to a subsequence, w_k converges weakly in H_0^1 and strongly in L^2 to some $w \in H_0^1$. Meanwhile, $-\Delta w_k$ tends to zero in $L^1_{loc}(\Omega)$ since $f(u_k) \leq au_k + b$ and l_k tends to ∞, this implies $-\Delta w = 0$ in Ω. Hence $w \equiv 0$, which is impossible because $\|w\|_2 = \lim_{k\to\infty} \|w_k\|_2 = 1$. So $\{u_k\}$ is bounded in $L^2(\Omega)$, hence in $H_0^1(\Omega)$ by equation. We prove readily that u^* is a solution of (P_{λ^*}) which contradicts (ii).

(iii) implies (ii). Any solution u of (P_{λ^*}) should satisfy $u \geq u_\lambda$, $\forall\, \lambda < \lambda^*$.

(ii)$\oplus$(iii) implies (i). Clearly $\lim_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Take $u_\lambda = l_\lambda w_\lambda$ with $\|w_\lambda\|_2 = 1$, then we have a subsequence w_k which converges weakly in H_0^1, strongly in L^2 and almost everywhere to $w \geq 0$. Moreover, in the sense of distribution,

$$-\Delta w = -\lim_{D'(\Omega)} \Delta w_k = \lim \frac{\lambda_k f(l_k w_k)}{l_k} = \lambda^* a w \geq \lambda_1 w \quad \text{a.e.}$$

Taking again φ_0 as test function, we see that the last inequality must be an equality, so $\lambda^* = \lambda_1/a$. $\square$

Remark that when $f(t) \geq at$ in $\mathbb{R}_+$, we cannot get a solution for $\lambda = \lambda_1/a$ since $f(t) > at$ in a neighborhood of 0 (using always φ_0 as test function), we obtain an important consequence of Theorem 2.6 and Proposition 2.5.

Corollary 2.7. — *If we have $\lim_{t\to\infty} f(t) - at = l \geq 0$, then $\lambda^* = \lambda_1/a$, and a unique solution u_λ exists for (P_λ) for $\lambda \in (0, \lambda^*)$ while no solution exists for $\lambda \geq \lambda^*$.*

Moreover, the following result is established in [**27**].

Proposition 2.8. — *If $\lim_{t\to\infty} f(t) - at = l < 0$, then $\lambda_1/a < \lambda^* < \lambda_1/r_0$. A unique solution $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ exists for (P_{λ^*}). Furthermore, for any $\lambda \in (\lambda_1/a, \lambda^*)$, we have a second solution v_λ for (P_λ), such that v_λ tends u.c. to ∞ in Ω when $\lambda \downarrow \lambda_1/a$.*