Central Extension of the Yangian Double

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Abstract

A central extension $\widehat{DY(\mathfrak{g})}$ of the double of the Yangian is defined for a simple Lie algebra $\mathfrak{g}$ with complete proof for $\mathfrak{g} = \mathfrak{sl}_2$. Basic representations and intertwining operators are constructed for $\widehat{DY(\mathfrak{sl}_2)}$.

Résumé

On définit une extension centrale $\widehat{DY(\mathfrak{g})}$ du double du Yangien pour une algèbre de Lie simple $\mathfrak{g}$ avec des preuves complètes pour le cas $\mathfrak{g} = \mathfrak{sl}_2$. On construit des représentations de base et des opérateurs d'entrelacement pour $\widehat{DY(\mathfrak{sl}_2)}$.

1 Introduction

The Yangian $Y(\mathbf{g})$ was introduced by V. Drinfeld [D1] as a Hopf algebra quantizing the rational solution $r(u) = \frac{e_i \otimes e^i}{u}$ of classical Yang-Baxter equation. As a Hopf algebra $Y(g)$ is a deformation of the universal enveloping algebra $U(\mathbf{g}[u])$ of polynomial currents to a simple Lie algebra $\mathbf{g}$ with respect to cobracket defined by $r(u)$. Unfortunately, up to the moment the representation theory of the Yangian is not so rich in applications as it takes place, for instance, for quantum affine algebras [JM]. We could mention the following gaps:

(i) The Yangian is not quasitriangular Hopf algebra, but pseudoquasitriangular Hopf algebra [D1];

(ii) There are no nontrivial examples of infinite-dimensional representations of $Y(\mathbf{g})$.

In order to get quasitriangular Hopf algebra one should introduce the quantum double $DY(\mathbf{g})$ of the Yangian. Detailed analisys of $DY(\mathbf{g})$ was done in [KT] together with explicit description of the universal R-matrix for $DY(\mathbf{g})$ (complete for $DY(sl_2)$ and partial in general case).

The most important examples of infinite dimensional representations of (quantum) affine algebras appear for nonzero value of central charge. Analogously, in the

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case of the Yangian we could expect the appearance of infinite dimensional representations only after central extension of $DY(\mathbf{g})$. This program is realized in this paper with complete proof for $DY(sl_2)$: we give a description of central extension $\widehat{DY(sl_2)}$ and construct its basic representations in bosonized form. In general case we present a description of $\widehat{DY(\mathbf{g})}$ without complete proof. In the forthcoming paper [KLP] we demonstrate, following general scheme of [DFJMN], how our construction produce the formulas for correlation functions in rational models [S].

The central extension of $DY(\mathbf{g})$ could be constructed in two ways. In Faddev--Reshetikhin-Takhtajan approach [FRT] we could describe $DY(\mathbf{g})$ by the set of equations

$$R^{\pm}_{V,W}(u-v)L^{\pm}_V(u)L^{\pm}_W(v) = L^{\pm}_W(v)L^{\pm}_V(u)R^{\pm}_{V,W}(u-v), \tag{1}$$

$$R^{+}_{V,W}(u-v)L^{+}_V(u)L^{-}_W(v) = L^{-}_W(v)L^{+}_V(u)R^{+}_{V,W}(u-v) \tag{2}$$

for matrix valued generating functions $L^{\pm}_V(u)$, $L^{\pm}_W(v)$ of $DY(\mathbf{g})$, where $V(u), W(v)$ are finite dimensional representations of $DY(\mathbf{g})$, $R^{\pm}_{V,W}$ are images of the universal R-matrix $\mathcal{R}$ and $(\mathcal{R}^{-1})^{21}$. Then, following [RS], we can make a shift of a spectral parameter in R-matrix in equation (2) by a central element:

$$R^{+}_{V,W}(u-v+c\hbar)L^{+}_V(u)L^{-}_W(v) = L^{-}_W(v)L^{+}_V(u)R^{+}_{V,W}(u-v). \tag{3}$$

For the construction of representations we should then extract Drinfeld generators of $\widehat{DY(\mathbf{g})}$ from L-operators (3) via their Gauss decomposition [DF].

We prefer another way, originaly used by V.Drinfeld [D2] for his "current" description of quantum affine algebras. The properties of comultiplication for $Y(\mathbf{g})$ show that one can extend the Yangian $Y(\mathbf{g})$ (or its dual with opposite comultiplication $Y^0(\mathbf{g})$) to a new Hopf algebra, adding the derivative d of a spectral parameter u. Alternativly, one can extend $Y(\mathbf{g})$ by automorphisms of shifts of u. The double of this extension is exactly what we want to find. Central element c is dual to derivative d.

The plan of the paper is as follows. First we remind the description of the Yangian $Y(sl_2)$ and of its quantum double from [KT]. Then we construct the central extension $\widehat{DY(sl_2)}$, describe the structure of the universal R-matrix for $\widehat{DY(sl_2)}$ and translate our description into L- operator language. In section 5 we construct basic representation of $\widehat{DY(sl_2)}$ and in the last section we describe the structure of $\widehat{DY(\mathbf{g})}$ in general case (without a proof).

2 $Y(sl_2)$ and its Quantum Double

The Yangian $Y(sl_2)$ can be described as a Hopf algebra generated by the elements e_k, h_k, f_k, $k \geq 0$ subjected to the relations

$$\begin{gathered}
[h_k, h_l] = 0 \ , \qquad [e_k, f_l] = h_{k+l} \ , \\
[h_0, e_l] = 2e_l \ , \qquad [h_0, f_l] = -2f_l \ , \\
[h_{k+1}, e_l] - [h_k, e_{l+1}] = \hbar\{h_k, e_l\} \ , \\
[h_{k+1}, f_l] - [h_k, f_{l+1}] = -\hbar\{h_k, f_l\} \ , \\
[e_{k+1}, e_l] - [e_k, e_{l+1}] = \hbar\{e_k, e_l\} \ , \\
[f_{k+1}, f_l] - [f_k, f_{l+1}] = -h\{f_k, f_l\} \ ,
\end{gathered} \tag{4}$$

where $\hbar$ is a parameter of the deformation, $\{a, b\} = ab + ba$. The comultiplication and the antipode are uniquely defined by the relations

$$\begin{gathered}
\Delta(e_0) = e_0 \otimes 1 + 1 \otimes e_0, \qquad \Delta(h_0) = h_0 \otimes 1 + 1 \otimes h_0, \qquad \Delta(f_0) = f_0 \otimes 1 + 1 \otimes f_0, \\
\Delta(e_1) = e_1 \otimes 1 + 1 \otimes e_1 + \hbar h_0 \otimes e_0, \qquad \Delta(f_1) = f_1 \otimes 1 + 1 \otimes f_1 + \hbar f_0 \otimes h_0, \\
\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1 + \hbar h_0 \otimes h_0 - 2\hbar f_0 \otimes e_0.
\end{gathered} \tag{5}$$

In terms of generating functions

$$e^+(u) := \sum_{k\geq 0} e_k u^{-k-1}, \qquad f^+(u) := \sum_{k\geq 0} f_k u^{-k-1}, \qquad h^+(u) := 1 + \hbar \sum_{k\geq 0} h_k u^{-k-1}$$

the relations (4) look as follows

$$\begin{gathered}
[h^+(u), h^+(v)] = 0 \ , \\
[e^+(u), f^+(v)] = -\frac{1}{\hbar}\frac{h^+(u) - h^+(v)}{u - v} \ , \\
[h^+(u), e^+(v)] = -\hbar\frac{\{h^+(u), (e^+(u) - e^+(v))\}}{u - v} \ , \\
[h^+(u), f^+(v)] = \hbar\frac{\{h^+(u), (f^+(u) - f^+(v))\}}{u - v} \ , \\
[e^+(u), e^+(v)] = -\hbar\frac{(e^+(u) - e^+(v))^2}{u - v} \ , \\
[f^+(u), f^+(v)] = \hbar\frac{(f^+(u) - f^+(v))^2}{u - v} \ .
\end{gathered} \tag{6}$$

The comultiplication is given by Molev's formulas [M], see also [KT]:

$$\Delta(e^+(u)) = e^+(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} \big(f^+(u + \hbar)\big)^k h^+(u) \otimes \big(e^+(u)\big)^{k+1}, \tag{7}$$

$$\Delta(f^+(u)) = 1\otimes f^+(u) + \sum_{k=0}^{\infty}(-1)^k\hbar^{2k}\big(f^+(u)\big)^{k+1}\otimes h^+(u)\big(e^+(u+\hbar)\big)^k \tag{8}$$

$$\Delta(h^+(u)) = \sum_{k=0}^{\infty}(-1)^k(k+1)\hbar^{2k}\big(f^+(u+\hbar)\big)^k h^+(u)\otimes h^+(u)\big(e^+(u+\hbar)\big)^k \tag{9}$$

Let now C be an algebra generated by the elements e_k, f_k, h_k, $(k\in \mathbf{Z})$, with relations (4). Algebra C admits $\mathbf{Z}$-filtration

$$\ldots\subset C_{-n}\subset\ldots\subset C_{-1}\subset C_0\subset C_1\ldots\subset C_n\ldots\subset C \tag{10}$$

defined by the conditions $\deg e_k=\deg f_k=\deg h_k=k$; $\deg x\in C_m\le m$. Let $\bar{C}$ be the corresponding formal completion of C. It is proved in [KT] that $DY(sl_2)$ is isomorphic to $\bar{C}$ as an algebra. In terms of generating functions

$$e^{\pm}(u):=\pm\sum_{\substack{k\ge 0\\ k<0}} e_k u^{-k-1},\qquad f^{\pm}(u):=\pm\sum_{\substack{k\ge 0\\ k<0}} f_k u^{-k-1},\qquad h^{\pm}(u):=1\pm\hbar\sum_{\substack{k\ge 0\\ k<0}} h_k u^{-k-1},$$

$$e(u)=e^+(u)-e^-(u),\qquad f(u)=f^+(u)-f^-(u)$$

the defining relations for $DY(sl_2)$ look as follows:

$$\begin{aligned}
h^a(u)h^b(v) &= h^b(v)h^a(u),\qquad a,b=\pm,\\
e(u)e(v) &= \frac{u-v+\hbar}{u-v-\hbar}\,e(v)e(u)\\
f(u)f(v) &= \frac{u-v-\hbar}{u-v+\hbar}\,f(v)f(u)\\
h^{\pm}(u)e(v) &= \frac{u-v+\hbar}{u-v-\hbar}\,e(v)h^{\pm}(u)\\
h^{\pm}(u)f(v) &= \frac{u-v-\hbar}{u-v+\hbar}\,f(v)h^{\pm}(u)\\
[e(u),f(v)] &= \frac{1}{\hbar}\delta(u-v)\left(h^+(u)-h^-(v)\right)
\end{aligned}\tag{11}$$

Here

$$\delta(u-v)=\sum_{n+m=-1}u^n v^m$$

satisfies the property

$$\delta(u-v)f(u)=\delta(u-v)f(v).$$

The comultiplication is given by the same formulas

$$\Delta(e^{\pm}(u)) = e^{\pm}(u)\otimes 1 + \sum_{k=0}^{\infty}(-1)^k\hbar^{2k}\big(f^{\pm}(u+\hbar)\big)^k h^{\pm}(u)\otimes \big(e^{\pm}(u)\big)^{k+1},$$

$$\Delta(f^{\pm}(u)) = 1\otimes f^{\pm}(u) + \sum_{k=0}^{\infty}(-1)^k\hbar^{2k}\big(f^{\pm}(u)\big)^{k+1}\otimes h^{\pm}(u)\big(e^{\pm}(u+\hbar)\big)^k$$

$$\Delta(h^{\pm}(u)) = \sum_{k=0}^{\infty}(-1)^k(k+1)\hbar^{2k}\big(f^{\pm}(u+\hbar)\big)^k h^{\pm}(u)\otimes h^{\pm}(u)\big(e^{\pm}(u+\hbar)\big)^k. \tag{12}$$

Subalgebra $Y^+ = Y(sl_2)\subset DY(sl_2)$ is generated by the components of $e^+(u)$, $f^+(u)$, $h^+(u)$ and its dual with opposite comultiplication $Y^- = (Y(sl_2))^0$ is a formal completion (see (10)) of subalgebra, generated by the components of $e^-(u)$, $f^-(u)$, $h^-(u)$.

Let us describe the Hopf pairing of Y^+ and Y^- [KT]. Note that by a Hopf pairing $<,>: A\otimes B\to \mathbf{C}$ we mean bilinear map satisfying the conditions

$$< a, b_1b_2 > = < \Delta(a), b_1\otimes b_2 >, \qquad < a_1a_2, b > = < a_2\otimes a_1, \Delta(b) > \tag{13}$$

The last condition is unusual but convenient in a work with quantum double.

Let $E^{\pm}$, $H^{\pm}$, $F^{\pm}$ be subalgebras (or their completions in Y^- case) generated by the components of $e^{\pm}(u)$, $h^{\pm}(u)$, $f^{\pm}(u)$. Subalgebras $E^{\pm}$ and $F^{\pm}$ do not contain the unit.

The first property of the Hopf pairing $Y^+\otimes Y^-\to\mathbf{C}$ is that it preserves the decompositions

$$Y^+ = E^+H^+F^+, \qquad Y^- = F^-H^-E^-.$$

It means that

$$< e^+h^+f^+, f^-h^-e^- > = < e^+, f^- >< h^+, h^- >< f^+, e^- > \tag{14}$$

for any elements $e^{\pm}\in E^{\pm}$, $h^{\pm}\in H^{\pm}$, $f^{\pm}\in F^{\pm}$. This property defines the pairing uniquely together with the relations

$$< e^+(u), f^-(v) > = \frac{1}{\hbar(u-v)}, \qquad < f^+(u), e^-(v) > = \frac{1}{\hbar(u-v)},$$

$$< h^+(u), h^-(v) > = \frac{u-v+\hbar}{u-v-\hbar}.$$

The full information of the pairing is encoded in the universal R-matrix for $DY(sl_2)$ which has the following form [KT]:

$$\mathcal{R} = R_+R_0R_- \ , \tag{15}$$