Coxeter Structure and Finite Group Action

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Abstract

Let $U(\mathfrak{g})$ be the enveloping algebra of a semi-simple Lie algebra $\mathfrak{g}$. Very little is known about the nature of $Aut\,U(\mathfrak{g})$. However, if G is a finite subgroup of $Aut\,U(\mathfrak{g})$ then very general results of Lorenz-Passman and of Montgomery can be used to relate $Spec\,U(\mathfrak{g})$ to $Spec\,U(\mathfrak{g})^G$. As noted by Alev-Polo one may read off the Dynkin diagram of $\mathfrak{g}$ from $Spec\,U(\mathfrak{g})$ and they used this to show that $U(\mathfrak{g})^G$ could not be again the enveloping algebra of a semi-simple Lie algebra unless G is trivial. Again let U be the minimal primitive quotient of $U(\mathfrak{g})$ admitting the trivial representation of $\mathfrak{g}$. A theorem of Polo asserts that if U^G is isomorphic to a similarly defined quotient of $U(\mathfrak{g}')$: $\mathfrak{g}'$ semi-simple, then $\mathfrak{g} \cong \mathfrak{g}'$. However in this case one cannot say that G is trivial.

The main content of this paper is the possible generalization of Polo's theorem to other minimal primitive quotients. A very significant technical difficulty arises from the Goldie rank of the almost minimal primitive quotients being > 1. Even under relatively strong hypotheses (regularity and integrality of the central character) one is only able to say that the Coxeter diagrams of $\mathfrak{g}$ and $\mathfrak{g}'$ coincide. The main thrust of the proofs is a systematic use of the Lorenz-Passman-Montgomery theory and the known very detailed description of $Prim\,U(\mathfrak{g})$. Unfortunately there is a severe lack of good examples. During this work some purely ring theoretic results involving Goldie rank comparisons and skew-field extensions are presented. A new inequality for Gelfand-Kirillov dimension is obtained and this leads to an interesting question involving a possible application of the intersection theorem.

Résumé

Soit $U(\mathfrak{g})$ l'algèbre enveloppante d'une algèbre de Lie semi-simple $\mathfrak{g}$. On sait très peu de choses sur $Aut\,U(\mathfrak{g})$. Néanmoins, si G désigne un sous-groupe fini de $Aut\,U(\mathfrak{g})$, alors des résultats généraux de Lorenz-Passman et Montgomery relient $Spec\,U(\mathfrak{g})$ à $Spec\,U(\mathfrak{g})^G$. Alev et Polo ont observé qu'on peut lire le diagramme de Dynkin de $\mathfrak{g}$ sur $Spec\,U(\mathfrak{g})$ et ils en ont déduit que $U(\mathfrak{g})^G$ ne peut être isomorphe à l'algèbre enveloppante d'une algèbre de Lie que si G est trivial. Soit U le quotient primitif minimal de $U(\mathfrak{g})$ admettant la représentation triviale de $\mathfrak{g}$. D'après un théorème de Polo, si U^G est isomorphe à un quotient

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de $U(\mathfrak{g}')$: $\mathfrak{g}'$ semi-simple, alors $\mathfrak{g} \cong \mathfrak{g}'$. Mais dans ce cas on ne peut affirmer que G est trivial.

Le contenu principal de ce papier est une possible généralisation du résultat de Polo à d'autres quotients primitifs minimaux. Une difficulté technique significative provient du fait que la dimension de Goldie peut alors être > 1. Même sous des hypothèses relativement fortes (régularité et intégralité du caractère central) on peut seulement dire que les diagrammes de Coxeter de $\mathfrak{g}$ et $\mathfrak{g}'$ coincident. Les preuves sont basées sur une utilisation systématique de la théorie de Lorenz-Passman et Montgomery et la connaissance très détaillée de $Prim\, U(\mathfrak{g})$. Malheureusement, il y a un manque sévère d'exemples. Dans ce travail, on présente quelques résultats de théorie des anneaux concernant des comparaisons de rangs de Goldie et des extensions de corps gauches. On obtient une nouvelle inégalité pour la dimension de Gelfand-Kirillov qui conduit à une question intéressante concernant une application du théorème d'intersection.

1 Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. Let G be a finite subgroup of Aut$U(\mathfrak{g})$. A remarkable recent result of J. Alev and P. Polo [AP, Thm.1] shows that $U(\mathfrak{g})^G$ cannot be again the enveloping algebra of some possibly different semisimple Lie algebra $\mathfrak{g}'$ unless G is trivial. Again let U_ρ (resp. V_ρ) be the minimal primitive quotient of $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}')$) admitting the trivial representation of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) and G a finite subgroup of AutU_ρ. Polo [P, Thm.7.1] has shown that if $U_\rho^G \cong V_\rho$ then $\mathfrak{g} \cong \mathfrak{g}'$.

1.2. The proof of the above results uses some general results on finite group actions (see Section 2) and some knowledge of Prim$U(\mathfrak{g})$. However the proofs are not particularly difficult and need relatively little from these two theories.

1.3. The aim of this paper is to generalize Polo's theorem to arbitrary (regular) central characters. At present the only interest for doing this is that the problem becomes very significantly harder and we need practically all that is known on the two theories discussed in 1.2. The obvious critique is that we know of no non-trivial examples of such finite group actions. Yet for example take $\mathfrak{g}$ of type B_2 (resp. G_2) with $\mathbb{Z}_2$ (resp. $\mathbb{Z}_3$) $\subset$ Aut$U(\mathfrak{g})$ acting via scalar multiplication on short root vectors. Then the maximal completely prime ideal P associated to the 4 (resp. 6) dimensional coadjoint orbit [J1] is $\mathbb{Z}_2$ (resp. is $\mathbb{Z}_3$) stable and "accidentally" the fixed subalgebra is isomorphic to a minimal primitive quotient of $U(\mathfrak{sl}(2) \times \mathfrak{sl}(2))$ (resp. $U(\mathfrak{sl}(3))$).

1.4. In Section 2 we review some general results on finite group actions and in particular the Montgomery bijection. In Section 3 we develop some comparison results on Goldie rank, particularly with respect to the additivity principle. In Section 4 we show that the isomorphism $U_\lambda^G \cong V_\mu$ (where λ, μ are dominant, regular elements of the appropriate Cartan subalgebras) implies that the (relative) Coxeter diagrams pertaining to U_λ and V_μ are isomorphic (Theorem 4.20). Unlike the situation encountered in the special case of Polo's theorem we are not able to say that G orbits in SpecU_λ are trivial (which also "trivializes" Montgomery's bijection). In Section 5 we relate λ, μ through an additivity principle (Theorem 5.16). However we are *not* able to say that the (relative) Dynkin diagrams pertaining to U_λ and V_μ are isomorphic. This question is examined in Section 6 where we show that it cannot be resolved by passage to rings of fractions and analysis of Goldie rank except in what we call the indivisible Goldie rank case (Theorem 6.7). This occurs for example in Polo's situation and leads to a refinement of that result. I would like to thank the referee for some remarks and corrections.

2 Finite Group Actions on Rings

2.1. Let B be a ring, G a finite subgroup of Aut$B$ and $A := B^G$ the fixed ring. A number of remarkable very general results relating Spec$B$ to Spec$A$ derive from the work of G. Bergman and I.M. Isaacs [BI], M. Lorenz and D.S. Passman [LP] and S. Montgomery [M2]. We detail what we need of this theory below. It will be assumed here and throughout this paper that $|G| \neq 0$ in B. We remark that in applications A, B are assumed noetherian and then the resulting weaker versions of these results partly go further back.

2.2. It is clear that G acts on Spec$B$ which is hence a disjoint union of G orbits which are in turn finite sets. Given $P \in$ Spec$B$ we denote by $O(P)$ the G orbit containing P. Then $I(P) := \bigcap_{P_i \in O(P)} P_i$, or simply I, is G invariant and so it is natural to consider I^G which is a semiprime ideal [BI] of A. Note however that $I^G = P_i^G = P_i \cap A$ for all $P_i \in O(P)$. If $P, P' \in$ Spec$B$ lie in the same G orbit we write $P \sim P'$. Obviously

Lemma — *The following are equivalent*

(i) $I(P) \supset I(P')$.

(ii) *For all* $P_i \in O(P)$ *there exists* $P'_j \in O(P')$ *such that* $P_i \supset P'_j$.

(iii) *For all* $P'_i \in O(P')$ *there exists* $P_j \in O(P)$ *such that* $P_j \supset P'_i$.

We write $O(P) \geq O(P')$ when one of these hold.

2.3. Define the group ring BG to be the free B module on generators $g \in G$ with multiplication $(bg, b'g') = (bg(b'), gg')$ where $b' \mapsto g(b')$ denotes the action of G on B.

Set $e = \frac{1}{|G|}\sum_{g \in G} g$ which is an idempotent of BG. After a classical result of Jacobson (see [LP, Lemma 4.5] for example) the map $\varphi : Q \mapsto eQe$ is an order isomorphism of $\{Q \in \text{Spec}BG \mid e \notin Q\}$ onto Spec$A$. Extend φ to a bijection of semiprime ideals. Define an equivalence relation $\sim$ on Spec$A$ by $p \sim p'$ if $\varphi^{-1}(p) \cap B = \varphi^{-1}(p') \cap B$ and let $C(p)$ denote the equivalence class containing p. Set $I(p) = \bigcap_{p' \in C(p)} p'$.

2.4. There are three key facts which lead to the Montgomery isomorphism [M2, Sect.3], namely

(i) For all $Q \in \mathrm{Spec}BG$ there exists $P \in \mathrm{Spec}B$ such that $Q \cap B = I(P)$.

(ii) $Q_i \in \mathrm{Spec}BG$ is minimal over $I(P)G \iff Q_i \cap B = I(P)$. Moreover there are finitely many such Q_i and $\bigcap_{i=1}^n Q_i = I(P)G$.

(iii) If an ideal J of BG strictly contains a prime Q then $J \cap B \supsetneq Q \cap B$.

With the exception of the very last statement of (ii), these are due in their most general form to Lorenz and Passman [LP, Lemma 4.2, Thm. 1.3, Thm. 1.2].

In (ii) choose $m \in \mathbb{N}$ and order the Q_i so that $e \in Q_i \iff i > m$. Taking $Q = \varphi^{-1}(p) : p \in \mathrm{Spec}A$ it follows from (ii) that $C(p) = \{\varphi(Q_i) : i \leq m\}$ and $\varphi^{-1}(p) \cap A = \bigcap_{i=1}^m \varphi(Q_i)$. In particular we note the

Lemma — *$C(p)$ is the set of minimal primes over $\varphi^{-1}(p) \cap A$.*

2.5. The above result immediately leads [M2, 3.5 (3)] to a partial analogue of 2.2 namely

Lemma — *The following are equivalent*

(i) $\varphi^{-1}(p) \cap B \supset \varphi^{-1}(p') \cap B$.

(ii) *For each $p_i \in C(p)$ there exists $p'_j \in C(p')$ such that $p_i \supset p'_j$.*

Proof. Assume (i). Then $\varphi^{-1}(p_i) \supset \bigcap_{j=1}^{n'} Q'_j$ and so p_i contains some $\varphi(Q'_j)$. □

We write $C(p) \geq C(p')$ when one of these hold. For our purposes it is a significant technical difficulty that we have no analogue of 2.2 (iii).

2.6. From 2.4 (i) and 2.4 (ii) one immediately obtains [M2, Thm. 3.4] the

Theorem — *The map $p \mapsto O(P)$, where $I(P) = \varphi^{-1}(p) \cap B$ factors to an order isomorphism Φ of* Spec$A/\sim$ *onto* Spec$B/\sim$.

2.7. It is clear that primes of Spec$B$ in the same G orbit have the same height. By 2.4 (iii) it follows [M2, Prop. 3.5] that equivalent primes of Spec$A$ are incomparable and have the same height. Moreover

Lemma — *One has $htp = htP$ given $P \in \Phi(p)$.*

2.8. Whilst $\varphi^{-1}(I(P)^G) = I(P)G$ it is not quite obvious if this implies that the inclusion $BI(P)^G B \subset I(P)$ is an equality. Fortunately we shall only need the

Lemma — *The minimal primes over $BI(P)^G B$ are the $P_i \in O(P)$.*

Proof. If P' is a minimal prime over $BI(P)^G B$ then so are its G translates and $I(P') \supset BI(P)^G B$. Consequently $I(P')^G \supset I(P)^G$. Then $I(P') \supset I(P)$ from 2.6 or directly from 2.4. Then P' contains some $P_i \in O(P)$. Conversely $BI(P)^G B \subset I(P)$ so P' is contained in some $P_j \in O(P)$. $\square$

3 Goldie Rank Comparison

3.1. We use some fairly standard methods to compare various Goldie ranks. In this we retain the hypotheses and notation of Section 2 except that B is assumed semisimple artinian. This implies (see for example [M1, Cor. 0.2 and Thm. 1.15]) the correspond ing properties for BG and then for A.

3.2. Observe that a left BG submodule of B is just a left ideal which is G stable.

Lemma — *A left ideal L of B is G stable (resp. and minimal) if and only if it takes the form Be with $e \in A$ idempotent (resp. and minimal).*

Proof. Let L' be a BG stable complement of L in B. Then e is obtained as the image of $1 \in B$ under the projection onto L defined by the decomposition $B = L \oplus L'$. Conversely right multiplication gives an algebra anti-isomorphism $A = B^G \xrightarrow{\sim} \mathrm{End}_{BG}B$ which restricts to an anti-isomorphism $K := eAe \xrightarrow{\sim} \mathrm{End}_{BG}Be$. Yet K is a skew-field if and only if e is minimal. $\square$

3.3. Let M be a left BG module.

Lemma — *Suppose $Ann_A M \in \mathrm{Spec}A$. Then the multiplication map $\theta : B \otimes_A M^G \to M$ is injective.*

Proof. Let e be a minimal idempotent of A such that $eM \neq 0$ and set $K = eAe$. The hypothesis on $Ann_A M$ implies that $B \otimes_A M^G = Be \otimes_K eM^G$.

Suppose $ker\theta \neq 0$. Since Be is a simple BG module by 3.2 and $\mathrm{End}_{BG}Be = K$ one may apply the Jacobson density theorem [H, Thm. 2.1.4] to obtain $m \in eM^G \setminus \{0\}$ for which $Be \otimes m \subset ker\ \theta$. Then $em = 0$ which is absurd. $\square$

3.4. Let M be a left BG module. One may give $B' := \mathrm{End}_B M$ a G-algebra structure through the action $\psi \mapsto g.\psi, \forall\ g \in G, \psi \in \mathrm{End}_B M$ by $(g.\psi)(m) = g(\psi(g^{-1}m)), \forall m \in M$. Then $g(\psi(bm)) = (g.\psi)(g(b)(gm))$. Set $A' = B'^G$. Then A', B' are also semisimple artinian rings.

Lemma — *Assume that A, B, A', B' are all simple and that $M^G \neq 0$. Then $rkB/rkA = rkB'/rkA'$.*

Proof. Take $m \in M^G \setminus \{0\}$. Then $Ann_B m$ is a BG submodule of B and so of the form Be' for some idempotent $e' \in A \setminus \{1\}$. Let $e \leq 1 - e'$ be a minimal idempotent of A. Since Be is a simple BG module by 3.2 we obtain an isomorphism $Be \xrightarrow{\sim} Bem$. In particular Bem is a simple BG submodule of M. Now $\mathrm{End}_B M = B'$ so $\mathrm{End}_{BG}M = B'^G = A'$. Since A' is assumed simple, it follows that M is an