Hyperbolic Equations in the Twentieth Century

Lars Gårding*

Abstract

The subject began with Huygens's theory of wave fronts as envelopes of smoother waves, and subsequent work by Euler, d'Alembert and Riemann. Singularities at the wave fronts were not understood before Hadamard's theory of "partie finie" at the beginning of this century. Contributions by Herglotz and Petrovsky and the theory of distributions created in the forties by Laurent Schwartz greatly illuminated the study of singularities of solutions of hyperbolic PDE's. Solutions of Cauchy's problem given by Hadamard, Schauder, Petrovsky, and the author are discussed. More recently, microlocal analysis, initiated by M. Sato and L. Hörmander led to important advances in understanding the propagation of singularities. Functional analysis together with distributions and microlocal analysis are expected to be useful well into the next century.

Résumé

Le sujet débute avec la théorie de Huygens qui considère les fronts d'onde comme des enveloppes d'ondes plus régulières, et se poursuit par les travaux de Euler, d'Alembert et Riemann. Les singularités des fronts d'onde n'ont pas été comprises avant la théorie de la « partie finie » de Hadamard au début de ce siècle. Les contributions de Herglotz, Petrovsky et dans les années quarante, la théorie des distributions de Laurent Schwartz ont éclairé l'étude des singularités des solutions des EDP hyperboliques. On passe en revue les solutions au problème de Cauchy

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*Dept. of Mathematics, Univ. of Lund, Box 118, S–22100 Lund, Sweden

données par Hadamard, Schauder, Petrovsky et l'auteur. Plus récemment, l'analyse microlocale de M. Sato et L. Hörmander a permis de grandes avancées dans la compréhension de la propagation des singularités. L'analyse fonctionnelle, les distributions et l'analyse microlocale seront certainement des outils importants du prochain siècle.

1. Introduction

The first example of a hyperbolic equation was the wave equation

$$u_{tt} - \Delta u = 0.$$

In one space variable n, the solutions describe free movements with velocity 1 in a perfectly elastic medium. A nonlinear version appears in one dimensional hydrodynamics. Riemann's 1860 treatment was later completed by the Rankine-Hugoniot jump conditions and conditions of entropy. Further examples of hyperbolic equations and systems appeared in the theory of electricity and magnetism and elasticity.

Originally, the adjective hyperbolic marked the connection between the wave equation and a hyperbolic conoid. When applied to general partial differential operators or systems the term now indicates that one of the variables is time $t = t(x)$ and that the solutions of the system describe wave propagation with finite velocity in all directions. More precisely, the solution u of Cauchy's problem with no source function and with data given for $t = \text{const.}$ should have the property that the value of u at a point depends continuously on the values of the data and their derivatives in a compact set. For an operator $P(D)$ with constant coefficients this means that there is a fundamental solution $E(x)$, i.e. a distribution such that $P(D)E(x) = \delta(x)$, whose support is contained in a proper, closed cone.

In the first half of the twentieth century, local existence by classical analysis of solutions to Cauchy's problem for hyperbolic equations with smooth data was the main problem. Soon after, functional analysis and distributions came into play and the introduction around 1970 of pseudodifferential operators and microlocal analysis of distributions was followed by a period of important results on the propagation of singularities, both free and under reflection in a boundary. Later this study was extended to nonlinear equations. Another question, latent during the period, is the problem of global existence of solutions for nonlinear equations close to linear ones. It took a new turn with the study of blow-up times by Fritz John.

Only a sample of the main results can be mentioned here. In particular, I refrain from the various hyperbolic aspects of hydrodynamics and the theory of scattering in spectral analysis.

The development of the theory of hyperbolic equations from 1900 cannot be understood without a review of some of the main results from the time before 1900. It is done here briefly under the heading of Prehistory.[1]

2. Prehistory

With three space variables the wave equation describes free propagation of light in physical space with velocity 1. For this equation, Poisson proved what in modern terms amounts to the fact that the wave operator $\Box = \partial_t^2 - \Delta$ has a fundamental solution

$$E(t,x) = \frac{1}{2\pi} H(t)\delta(t^2 - |x|^2)$$

with support on the forward lightcone $t = |x|$. It was then only too easy to believe this to be a general phenomenon, for instance that the equations for the propagation of light in media with double refraction follow the same rule known under the name of Huygens principle:[2] *all* light from a point-source is concentrated to the surface given by the rules of geometric optics. Both G. Lamé and Sonya Kovalevski made this mistake till the use of Fourier analysis proved that the existence of diffuse light outside such surfaces is the rule and the contrary an exception (for a historical review, see [Gårding 1989]).

A fundamental solution of the wave operator for two space variables was found by Volterra and, at the turn of the century, Tedone tried the general case, but could only construct what amounts to sufficiently repeated integrals with respect to time of purported fundamental solutions. Behind these difficulties is the fact that, in contrast to the properties of Laplace's operator, the fundamental solutions of the wave operator are distributions with singularities outside the pole which get worse as the number n of space variables increases. Before the theory of distributions, this was a formidable difficulty.

3. "Partie finie"

The obstacle which stopped Tedone, was surmounted by Hadamard in his theory of *partie finie*, found before 1920 and exposed in [Hadamard 1932].

[1]The remarks and notes of Hadamard's book 1932give a fuller account.

[2]Huygens's minor premise according to Hadamard [1932].

His operator is the wave operator with smooth, variable coefficients and has the form

$$L(x, \partial_x) = \sum a_{jk}(x)\partial_j\partial_k + \text{lower terms} \tag{3.1}$$

where the metric form $\sum a_{jk}\xi_j\xi_k$ has Lorentz signature $+, -\ldots-$. A direction for which the inverse metric form is positive, zero or negative is said to be time-like, light-like and space-like respectively. Surfaces with time-like and space-like normals are said to be space-like and time-like respectively. The light rays are the geodesics of length zero. A time function $t(x)$ with $t'(x)$ time-like is given.

The light rays with a positive time direction issued from a point y constitute the forward light cone C_y with its vertex at y. Inside this light cone, the fundamental solution with its pole at y has the same form as in the elliptic case

$$f(x, y)d(x, y)^{2-n} \tag{3.2}$$

where f is a smooth function and d is the geodesic distance between x and y. The difficulty is that $d(x, y) = 0$ when $x \in C_y$. The *partie finie* can be said to be a renormalization procedure which extends this formula for n odd to a distribution which is also a fundamental solution. For n even, Hadamard uses what is called the method of descent. In the work by M. Riesz [1949] the exponent $2 - n$ of (3.2) is replaced by $\alpha - n$ where α is a complex paramater. At the same time f is made to depend on α and a denominator $\Gamma(\alpha/2)\Gamma((\alpha + 2 - n)/2)$ is introduced. The stage is then set for an analytical continuation with respect to α. In this way and for selfadjoint operators L, Riesz constructs kernels of the complex powers of L.

In his case, Hadamard could give a complete local solution of Cauchy's problem with data on a space-like surface, but the corresponding mixed problem with reflection in a time-like surface presented insurmountable difficulties.

4. Friedrichs-Lewy energy density, existence proofs by Schauder and Petrovsky

The discovery of Friedrichs and Lewy [1928] that $\partial_1 u \Box u$ with u real is the divergence of a tensor with a positive energy density on space-like surfaces produced both uniqueness results and *a priori* energy estimates, decisive for the later development.

A great step forward was taken by Schauder [1935, 1936a,b] who proved local existence of solutions of Cauchy's problem and also the mixed problem for quasilinear wave operators. The method is to use approximations starting from the case of analytic coefficients and analytic data. The success of these papers depends on stable energy estimates derived from the energy tensor and the use of the fact that square integrable functions with square integrable derivatives up to order n form a ring under multiplication.[3]

Only a year after Schauder, Petrovsky [1937] extended his results for Cauchy's problem to strongly hyperbolic systems, in the simplest case

$$u_t + \sum_1^n A_k(t,x)u_k + Bu = v, \quad u_k = \partial u/\partial x_k, \tag{4.1}$$

and the corresponding quasilinear versions. Here the coefficients are square matrices of order m and the strong hyperbolicity with respect to the time variable t means that all m velocities c given by

$$\det(cI + \sum \xi_k A_k(t,x)) = 0 \tag{4.2}$$

are real and separate for all real $\xi \neq 0$. The method is that of Schauder starting from the analytic case, but Petrovsky had to find his own energy estimate. For this he used the Fourier transform, but the essential point is to be found in thirty rather impenetrable pages. Note that if the system (4.1) is symmetric, *i.e.*, the matrices A_k are Hermitian symmetric, then (4.2) holds except that the velocities need not be separate. Moreover,

$$\partial_t |u(t,x)|^2 + \sum \partial_k (A_k u(t,x), u(t,x)) = O(|u(t,x)|^2 + |u(t,x)||v(t,x)|)$$

under suitable conditions on the coefficients. Hence the proper energy density on $t = \text{const}$ is here simply $|u(t,x)|^2 dx$.

Petrovsky's paper was followed by a study [Petrovsky 1938] of conditions for the continuity of Cauchy's problem for operators whose coefficients depend only on time.

5. Fundamental solutions, Herglotz and Petrovsky

Herglotz [1926-28] and Petrovsky [1945] used the Fourier transform to construct fundamental solutions $E(P,t,x)$ for constant coefficient homogeneous differential operators $P = P(\partial_t, \partial_x)$ of degree m which are strongly hyperbolic with respect to t. Every such fundamental solution E is analytic outside a

[3]Soon after, Sobolev proved that one gets a ring also when n is replaced by $(n+1)/2$ when n is odd and by $(n+2)/2$ when n is even.