From General Relativity to Group Representations

The Background to Weyl's Papers of 1925–26

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Abstract

Hermann Weyl's papers on the representation of semisimple Lie groups (1925-26) stand out as landmarks of twentieth century mathematics. The following essay focuses on how Weyl came to write these papers. It offers a reconstruction of his intellectual journey from intense involvement with the mathematics of general relativity to that of the representation of groups. In particular it calls attention to a 1924 paper by Weyl on tensor symmetries that played a pivotal role in redirecting his research interests. The picture that emerges illustrates how Weyl's broad philosophically inclined interests inspired and informed his creative work in pure mathematics.

Résumé

Les articles de Hermann Weyl sur la représentation des groupes de Lie semi-simples (1925-26) apparaissent comme des étapes majeures des mathématiques du vingtième siècle. En analysant ce qui a amené Weyl à écrire ces articles, cet essai présente une reconstruction de sa démarche intellectuelle, depuis les mathématiques de la relativité générale jusqu'à celles des représentations de groupes. Il attire notamment l'attention sur l'article de 1924 sur les symétries tensorielles, pivot de la réorientation de ses domaines de recherche. On voit aussi comment les larges intérêts et les motivations philosophiques de Weyl ont inspiré et enrichi sa créativité en mathématiques pures.

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Dieudonné once wrote that "progress in mathematics results, most of the time, through the imaginative fusion of two or more apparently different topics" [Dieudonné 1975, p. 537]. One of the most brilliant examples of progress by fusion is provided by Herman Weyl's celebrated papers on the representation of semisimple Lie groups (1925-1926). For in them he fashioned a theory which embraced I. Schur's recent work (1924) on the invariants and representations of the n-dimensional rotation group, which was conceived within the conceptual framework of Frobenius' theory of group characters and representations, and E. Cartan's earlier work (1894–1913) on semisimple Lie algebras, which was done within the framework of Lie's theory of groups and had been unknown to Schur. Moreover, in fashioning his theory of semisimple groups, Weyl drew on a host of ideas from such historically disparate areas as Frobenius' theory of finite group characters, Lie's theory, tensor algebra, invariant theory, complex function theory (Riemann surfaces), topology and Hilbert's theory of integral equations. Weyl's papers were thus a paradigm of fusion, and they exerted a considerable influence on subsequent developments. They stand out as one of the landmarks of twentieth century mathematics.

It is not my purpose here to describe the rich contents of these remarkable papers nor to analyze their influence. This has been done by Chevalley and Weil [1957], by Dieudonné [1976], and, above all, by Borel [1986]. I wish to focus instead on how Weyl came to write these remarkable papers. In this connection Weyl wrote:

> "for myself I can say that the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups ..."[Weyl 1949, p. 400].

My goal is to attempt to explain what Weyl meant by this remark, that is, to reconstruct the historical picture of his intellectual journey from his involvement with the mathematics of general relativity to that of the representation of semisimple Lie groups. In particular, I want to call attention to a paper by Weyl [1924a], which in my opinion adds a fullness and clarity to the picture that would otherwise be lacking. The picture that emerges illustrates how Weyl's broad philosophically inclined interests — in this instance in theoretical physics — inspired and informed his creative work in pure mathematics.[1]

[1]For another such instance, see [Scholz 1995] where Weyl's interest in Fichte's philosophy is related to his approach to the geometry of manifolds.

The Space Problem

Weyl's involvement with general relativity began in 1916, when, at age 31, he returned from military service to his position at the Eidgenössische Technische Hochschule (ETH) in Zürich. "My mathematical mind was as blank as any veteran's," he later recalled, "and I did not know what to do. I began to study algebraic surfaces; but before I had gotten far, Einstein's memoir came into my hand and set me afire."[2] By the summer of 1917 Weyl was lecturing on general relativity at the ETH. These lectures formed the starting point for his classic book, *Raum, Zeit, Materie*, which went through four editions during 1918–23,[3] and spawned many collateral publications by Weyl aimed at further developing the ideas and implications of his lectures. One of the outcomes of Weyl's reflections on general relativity was his introduction of what he called a "purely infinitesimal geometry."[4]

Weyl became convinced that Riemannian geometry, including the quasi Riemannian geometry of an indefinite metric $ds^2 = \sum_{ij} g_{ij} dx_i dx_j$, $g_{ij} = g_{ij}(x_1, \ldots, x_n)$, on which Einstein's theory was based, was not a consistently infinitesimal geometry. That is, in Riemannian geometry, a vector $v = (dx_1, \ldots, dx_n)$ in the tangent plane at point P of the manifold could only be compared with a vector $w = (dy_1, \ldots, dy_n)$ in the tangent plane at point Q in the relative sense of a path-dependent parallel transport from P to Q, but the lengths of v and w were absolutely comparable in the sense that

$$\frac{|v|}{|w|} = \sqrt{\frac{\sum_{i,j} g_{ij}(P) dx_i dx_j}{\sum_{i,j} g_{ij}(Q) dy_i dy_j}}.$$

These considerations led Weyl to a generalization of Riemannian geometries in which the lengths of v and w are not absolutely comparable. As in Riemannian geometry a nondegenerate quadratic differential form ds^2 of constant signature is postulated but metric relations are determined locally only up to a positive calibration (or gauge) factor λ and so are given by $ds^2 = \sum_{ij} \lambda g_{ij} dx_i dx_j$. Here λ varies from point to point in such a way that the comparison of the lengths of v at P and w at Q is also in general a path-dependent process.[5]

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[2] Quoted by S. Sigurdsson [1991, p. 62] from Weyl's unpublished "Lecture at the Bicentennial Conference" (in Princeton).

[3] There were actually five editions, but the second (1919) was simply a reprint of the first [Scholz 1994, p. 205n].

[4] See Scholz [1994, 1995] for a detailed account of the historical context and evolution of Weyl's ideas on this theory during 1917–23 .

[5] For a complete definition of Weyl's geometry see [Scholz 1994, p. 213] and for a contemporary formulation see [Folland 1970]. Weyl's geometry represented the first of a succession of gauge theories that has continued into present-day physics [Vizgin 1989, p. 310].

Although Weyl's geometry was motivated by the above critique of Riemannian geometry, he discovered that he could use its framework to develop a unified field theory, that is, a theory embracing both the gravitational and the electromagnetic field. Hilbert had been the first to devise a unified theory within the framework of general relativity in 1915. Weyl's theory was presented in several papers during 1918–19 and in the third edition (1919) of *Raum, Zeit, Materie*. Einstein admired Weyl's theory for its mathematical brilliance, but he rejected it as physically impossible. Although Weyl respected Einstein's profound physical intuition and was accordingly disappointed by the negative reaction to his unified theory, Einstein's arguments did not convince him that his own approach was wrong. His belief in the correctness of his theory was bolstered by the outcome of his reconsideration, in publications during 1921–23, of the "space problem" first posed by Helmholtz in 1866. It was in connection with this problem that Weyl first began to appreciate the value of group theory for investigating questions of interest to him involving the mathematical foundations of physical theories.

In 1866 Helmholtz sought to deduce the geometrical properties of space from facts about the existence and motion of rigid bodies. He concluded that the distance between points (x, y, z) and $(x + dx, y + dy, z + dz)$ is $\sqrt{dx^2 + dy^2 + dz^2}$ and that space is indeed Euclidean. He returned to the matter in 1868, however, after learning from the work of Riemann and Beltrami about geometries of constant curvature. Using the properties of rigid bodies he had singled out earlier, he argued that Riemann's hypothesis that metric relations are given locally by a quadratic differential form is actually a mathematical consequence of these facts. Later, in 1887, Poincaré obtained Helmoltz's results for two-dimensional space by applying Lie's theory of groups and utilizing, in particular, the consideration of Lie algebras. Lie himself considered the problem in n dimensions by means of the consideration of Lie groups and algebras in 1892. The Lie-Helmholtz treatment of the space problem, however, was rendered obsolete by the advent of general relativity since, as Weyl put it:

> "Now we are ... dealing with a four-dimensional [continuum] with a metric based not on a positive definite quadratic form but rather one that is indefinite. What is more, we no longer believe in the metric homogeneity of this medium — the very foundation of the Helmholtzian metric — since the metric field is not something fixed but rather stands in causal dependency on matter" [Weyl 1921a, p. 263].

Following the Helmholtz-Lie tradition, Weyl conceived of space (including therewith the possibility of space-time) as an n-dimensional differentiable manifold $\mathcal{M}$ with metric relations determined by the properties of congruences which are conceived in terms of groups. Thus at each point $P \in \mathcal{M}$ the rotations at P are assumed to form a continuous group of linear transformations $\mathfrak{G}_P$, and since the volume of parallelepipeds is assumed to be preserved by rotations, the $\mathfrak{G}_P$ are taken as subgroups of $\mathbf{SL}\ (T_P(\mathcal{M}))$. Metrical relations in a neighborhood $\mathcal{U}$ of P are then based on the assumption that all rotations at $P' \in \mathcal{U}$ can be obtained from a single linear congruence transformation A taking P to P' by composition with the rotations at P; that is, every $T' \in \mathfrak{G}_{P'}$ is of the form $T' = ATA^{-1}$ so that $\mathfrak{G}_{P'} = A\mathfrak{G}_P A^{-1}$. By "passing continuously" from P to any point Q of the manifold $\mathcal{M}$, Weyl was led to the assumption that all the groups $\mathfrak{G}_P$ are congruent to a group $\mathfrak{G} \subset \mathbf{SL}(n)$ with Lie algebra $\mathfrak{g} \subset \mathbf{sl}(n)$. Thus, whereas in the Lie-Helmholtz treatment of the space problem, the homogeneity of space entails the identity of the rotation groups at diverse points, in Weyl's formulation the rotation groups have differing "orientations," although they share the same abstract Lie algebra.

Within this mathematical context Weyl stipulated two postulates: (1) the nature of space imposes no restriction on the metrical relationship; (2) the affine connection is uniquely determined by the metrical relationship. His interesting mathematical interpretation of these two postulates led to the conclusion that the Lie algebra $\mathfrak{g}$ must possess the following properties:
a) For all $X \in \mathfrak{g}$, tr $X = 0$ (*i.e.*, $\mathfrak{g} \subset \mathbf{sl}(n, \mathbb{R})$);
b) dim $\mathfrak{g} = \frac{1}{2}n(n-1)$;
c) For any $X_1, \ldots, X_n \in \mathfrak{g}$ with matrix form $X_k = (a_{ij}^{(k)})$ with regard to some basis, if Col i of $X_j =$ Col j of X_i for all $i, j = 1, ..., n$, then $X_i = 0$ for all $i = 1, ..., n$.

In the fourth edition of *Raum, Zeit, Materie*, where Weyl first presented his analysis of the space problem [Weyl 1921a, §18], he pointed out that the Lie algebras $\mathfrak{g}_Q$ of all orthogonal groups with respect to a nonsingular quadratic form Q satisfy (a)–(c) and he conjectured as "highly probable" the following theorem which he had confirmed for $n = 2, 3$:

Theorem 1. — *The only Lie algebras satisfying (a)–(c) are the orthogonal Lie algebras $\mathfrak{g}_Q$ corresponding to a nondegenerate quadratic form Q.*

Weyl's conjectured theorem thus implied the locally Pythagorean nature of space. Weyl pointed out that when $\mathfrak{g}$ does correspond to an orthogonal Lie algebra, the quadratic form Q is only determined up to a constant of proportionality [Weyl 1921a, p. 146]. Although he did not say it explicitly at