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TRACTOR BUNDLES FOR IRREDUCIBLE PARABOLIC GEOMETRIES

by

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Abstract. — We use general results on tractor calculi for parabolic geometries that we obtained in a previous article to give a simple and effective characterisation of arbitrary normal tractor bundles on manifolds equipped with an irreducible parabolic geometry (also called almost Hermitian symmetric– or AHS–structure in the literature). Moreover, we also construct the corresponding normal adjoint tractor bundle and give explicit formulae for the normal tractor connections as well as the fundamental D–operators on such bundles. For such structures, part of this information is equivalent to giving the canonical Cartan connection. However it also provides all the information necessary for building up the invariant tractor calculus. As an application, we give a new simple construction of the standard tractor bundle in conformal geometry, which immediately leads to several elements of tractor calculus.

***Résumé* (Fibrés des tracteurs pour des géométries paraboliques irréductibles)**

Nous utilisons les résultats sur les calculs tractoriels pour des géométries paraboliques, obtenus dans un article précédent, afin de donner une caractérisation simple et effective pour des fibrés des tracteurs normaux arbitraires sur des variétés munies d'une géométrie parabolique irréductible (appelée également dans la littérature structure presque hermitienne symétrique). De plus, on construit le fibré des tracteurs normal associé et on donne des formules explicites pour les connexions sur le fibré de tracteurs normal et pour le D–opérateur fondamental sur de tels fibrés. Pour de telles structures, une partie de cette information est équivalente à la donnée de la connexion de Cartan canonique. Néanmoins, elle donne également toute l'information nécessaire pour construire le calcul invariant des tracteurs. Comme application, on donne une nouvelle construction simple du fibré des tracteurs standard en géométrie conforme, qui donne lieu immédiatement à plusieurs éléments de calculs tractoriels.

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1. Tractor bundles and normal tractor connections

Riemannian and pseudo-Riemannian geometries are equipped with a canonical metric and the metric (or Levi-Civita) connection that it determines. For this reason, in the setting of these geometries, it is natural to calculate directly with the tangent bundle, its dual and the tensor bundles. On the other hand for many other interesting structures such as conformal geometries, CR geometries, projective geometries and quaternionic structures the situation is not so fortunate. These structures are among the broad class of so-called parabolic geometries and for the geometries within this class there is no canonical connection or metric on the tangent bundle or the tensor bundles. Nevertheless for these structures there is a class of natural vector bundles which do have a canonical connection. These are the tractor bundles and the calculus based around these bundles is a natural analog of the tensor bundle and Levi-Civita connection calculus of Riemannian geometry.

Tractor calculus has its origins in the work of T.Y. Thomas [**11**] who developed key elements of the theory for conformal and projective geometries. Far more recently this was rediscovered and extended in [**1**]. Since this last work tractor calculus has been further extended and developed and the structures treated explicitly include CR and the almost Grassmannian/quaternionic geometries (see for example [**6, 7, 8, 9**] and references therein). Included in these works are many applications to the construction of invariant operators and polynomial invariants of the structures.

In our recent paper [**3**] we have introduced the concepts of tractor bundles and normal tractor connections for all parabolic geometries. Besides showing that from these bundles one can recover the Cartan bundle and the normal Cartan connection of such a geometry, we have also developed an invariant calculus based on adjoint tractor bundles and the so–called fundamental D–operators for all these geometries. Moreover, in that paper a general construction of the normal adjoint tractor bundle in the case of irreducible parabolic geometries is presented. While this approach, based on the adjoint representation of the underlying Lie–algebra, has the advantage of working for all irreducible parabolic geometries simultaneously, there are actually simpler tractor bundles available for each concrete choice of the structure. In fact, all previously known examples of tractor calculi as mentioned above are of the latter type. It is thus important to be able to recognise general normal tractor bundles for a parabolic geometry and to find the corresponding normal tractor connections.

The main result of this paper is theorem 1.3 which offers a complete solution for the case of irreducible parabolic geometries. For a given structure and representation of the underlying Lie algebra, this gives a characterisation of the normal tractor bundle, as well as a univsersal formula for the normal tractor connection. On the one hand this may be used to identify a bundle as the normal tractor bundle and then compute the normal tractor connection. On the other hand the theorem specifies the necessary ingredients for the construction of such a bundle. It should be pointed out, that the results obtained here are independent of the construction of the normal adjoint tractor bundles for irreducible parabolic geometries given in [**3**]. From that source we only use the technical background on these structures.

We will show the power of this approach in section 2 and 3 by giving an alternative construction of the most well known example of a normal tractor bundle, namely the standard tractors in conformal geometry. Besides providing a short and simple route to all the basic elements of conformal tractor calculus, this new construction also immediately encodes some more advanced elements of tractor calculus.

1.1. Background on irreducible parabolic geometries. — Parabolic geometries may be viewed as curved analogs of homogeneous spaces of the form G/P, where G is a real or complex simple Lie group and $P \subset G$ is a parabolic subgroup. In general, a parabolic geometry of type (G, P) on a smooth manifold M is defined as a principal P–bundle over M, which is endowed with a Cartan connection, whose curvature satisfies a certain normalization condition. This kind of definition is however very unsatisfactory for our purposes. The point about this is that these normal Cartan connections usually are obtained from underlying structures via fairly complicated prolongation procedures, see e.g. [**4**]. Tractor bundles and connections are an alternative approach to these structures, which do not require knowledge of the Cartan connection but may be constructed directly from underlying structures in many cases. Hence, in this paper we will rather focus on the underlying structures and avoid the general point of view via Cartan connections.

Fortunately, these underlying structures are particularly easy to understand for the subclass of irreducible parabolic geometries, which correspond to certain maximal parabolics. The point is that for these structures, one always has a (classical first order) G_0–structure (for a certain subgroup $G_0 \subset G$) on M, as well as a class of preferred connections on the tangent bundle TM. While both these are there for any irreducible parabolic geometry, their role in describing the structure may vary a lot, as can be seen from two important examples, namely conformal and classical projective structures.

In the conformal case, the G_0–structure just is the conformal structure, i.e. the reduction of the frame bundle to the conformal group, so this contains all the information. The preferred connections are then simply all torsion free connections respecting the conformal structure, i.e. all Weyl connections. On the other hand, in the projective case, the group G_0 turns out to be a full general linear group, so the first order G_0–structure contains no information at all, while the projective structure is given by the choice of a class of preferred torsion free connections.

The basic input to specify an irreducible parabolic geometry is a simple real Lie group G together with a so–called $|1|$–grading on its Lie algebra $\mathfrak{g}$, i.e. a grading of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. It is then known in general (see e.g. [**12**, section 3]) that $\mathfrak{g}_0$ is a reductive Lie algebra with one dimensional centre and the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$

is irreducible (which is the reason for the name "irreducible parabolic geometries"). Moreover, any $\mathfrak{g}$–invariant bilinear form (for example the Killing form) induces a duality of $\mathfrak{g}_0$–modules between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. Next, there is a canonical generator E, called the *grading element*, of the centre of $\mathfrak{g}_0$, which is characterised by the fact that its adjoint action on $\mathfrak{g}_j$ is given by multiplication by j for $j = -1, 0, 1$.

Having given these data, we define subgroups $G_0 \subset P \subset G$ by

$$G_0 = \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } i\}$$
$$P = \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \text{ for } i = 0, 1\},$$

where Ad denotes the adjoint action and we agree that $\mathfrak{g}_i = \{0\}$ for $|i| > 1$. It is easy to see that G_0 has Lie algebra $\mathfrak{g}_0$, while P has Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. An important result is that P is actually the semidirect product of G_0 and a vector group. More precisely, one proves (see e.g. [**4**, proposition 2.10]) that for any element $g \in P$ there are unique elements $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$ such that $g = g_0 \exp(Z)$. Hence if we define $P_+ \subset P$ as the image of $\mathfrak{g}_1$ under the exponential map, then $\exp : \mathfrak{g}_1 \to P_+$ is a diffeomorphism and P is the semidirect product of G_0 and P_+.

If neither $\mathfrak{g}$ nor its complexification is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$ with the $|1|$–grading given in block form by $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$, where the blocks are of size 1 and $n-1$, then a parabolic geometry of type (G, P) on a smooth manifold M (of the same dimension as $\mathfrak{g}_{-1}$) is defined to be a first order G_0–structure on the manifold M, where G_0 is viewed as a subgroup of $\mathrm{GL}(\mathfrak{g}_{-1})$ via the adjoint action. We will henceforth refer to these structures as the structures which are not of projective type.

On the other hand, if either $\mathfrak{g}$ or its complexification is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$ with the above grading, then this is some type of a projective structure, which is given by a choice of a class of affine connections on M (details below). See [**5**, 3.3] for a discussion of various examples of irreducible parabolic geometries.

Given a $|1|$–graded Lie algebra $\mathfrak{g}$, the simplest choice of group is $G = \mathrm{Aut}(\mathfrak{g})$, the group of all automorphisms of the Lie algebra $\mathfrak{g}$. Note that, for this choice of the group G, P is exactly the group $\mathrm{Aut}_f(\mathfrak{g})$ of all automorphism of the *filtered* Lie algebra $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{g}_1$, while G_0 is exactly the group $\mathrm{Aut}_{gr}(\mathfrak{g})$ of all automorphisms of the *graded* Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. For a general choice of G, the adjoint action shows that P (respectively G_0) is a covering of a subgroup of $\mathrm{Aut}_f(\mathfrak{g})$ (respectively $\mathrm{Aut}_{gr}(\mathfrak{g})$) which contains the connected component of the identity. Note however, that in any case the group P_+ is exactly the group of those automorphisms φ of $\mathfrak{g}$ such that for each $i = -1, 0, 1$ and each $A \in \mathfrak{g}_i$ the image $\varphi(A)$ is congruent to A modulo $\mathfrak{g}_{i+1} \oplus \mathfrak{g}_{i+2}$.

In any case, as shown in [**3**, 4.2, 4.4], on any manifold M equipped with a parabolic geometry of type (G, P) one has the following basic data:

(1) A principal G_0–bundle $p : \mathcal{G}_0 \to M$ which defines a first order G_0–structure on M. (In the non–projective cases, this defines the structure, while in the projective

cases it is a full first order frame bundle.) The tangent bundle TM and the cotangent bundle T^*M are the associated bundles to $\mathcal{G}_0$ corresponding to the adjoint action of G_0 on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, respectively. There is an induced bundle $\operatorname{End}_0 TM$ which is associated to $\mathcal{G}_0$ via the adjoint action of G_0 on $\mathfrak{g}_0$. This is canonically a subbundle of $T^*M \otimes TM$ and so we can view sections of this bundle either as endomorphisms of TM or of T^*M.
(2) An algebraic bracket $\{\ ,\ \} : TM \otimes T^*M \to \operatorname{End}_0 TM$, which together with the trivial brackets on $TM \otimes TM$ and on $T^*M \otimes T^*M$, the brackets $\operatorname{End}_0 TM \otimes TM \to TM$ given by $\{\Phi, \xi\} = \Phi(\xi)$ and $\operatorname{End}_0 TM \otimes T^*M \to T^*M$ given by $\{\Phi, \omega\} = -\Phi(\omega)$, and the bracket on $\operatorname{End}_0 TM \otimes \operatorname{End}_0 TM \to \operatorname{End}_0 TM$ given by the commutator of endomorphisms of TM, makes $T_xM \oplus \operatorname{End}_0 T_xM \oplus T_x^*M$, for each point $x \in M$, into a graded Lie algebra isomorphic to $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. (This algebraic bracket is induced from the Lie algebra bracket of $\mathfrak{g}$.)
(3) A preferred class of affine connections on M induced from principal connections on $\mathcal{G}_0$, such that for two preferred connections ∇ and $\hat{\nabla}$ there is a unique smooth one–form $\Upsilon \in \Omega^1(M)$ such that $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \{\{\Upsilon, \xi\}, \eta\}$ for all vector fields ξ, η on M. (In the projective cases, the structure is defined by the choice of this class of connections, while in the non–projective cases their existence is a nontrivial but elementary result.) Moreover, there is a restriction on the torsion of preferred connections, see below.

There is a nice reinterpretation of (1) and (2): Define the bundle $\overrightarrow{\mathcal{A}} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 \to M$ by $\mathcal{A}_{-1} = TM$, $\mathcal{A}_0 = \operatorname{End}_0 TM$ and $\mathcal{A}_1 = T^*M$. Then the algebraic bracket from (2) makes $\overrightarrow{\mathcal{A}}$ into a bundle of graded Lie algebras. Moreover, since $\mathcal{A}_i$ is the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$ the definition of the algebraic bracket implies that each point $u_0 \in \mathcal{G}_0$ lying over $x \in M$ leads to an isomorphism $\underline{u_0} : \mathfrak{g} \to \mathcal{A}_x$ of graded Lie algebras. In this picture, the principal right action of G_0 on $\mathcal{G}_0$ leads to $\underline{u_0 \cdot g} = \underline{u_0} \circ \operatorname{Ad}(g)$.

There are a few important facts on preferred connections that have to be noted. First, since they are induced from principal connections on $\mathcal{G}_0$, the algebraic brackets from (2) are covariantly constant with respect to any of the preferred connections. Second, the Jacobi identity immediately implies that $\{\{\Upsilon, \xi\}, \eta\}$ is symmetric in ξ and η, so all preferred connections have the same torsion $T \in \Gamma(\Lambda^2 T^*M \otimes TM)$. Hence, this torsion is an invariant of the parabolic geometry. The normalisation condition on the torsion mentioned above is that the trace over the last two entries of the map $\Lambda^2 TM \otimes T^*M \to \operatorname{End}_0 TM$ defined by $(\xi, \eta, \omega) \mapsto \{T(\xi, \eta), \omega\}$ vanishes. That is, in the language of [**3**], the torsion is ∂^*–closed.

There are also a few facts on the curvature of preferred connections that we will need in the sequel: Namely, if ∇ is a preferred connection, and $R \in \Gamma(\Lambda^2 T^*M \otimes \operatorname{End}_0 TM)$ is its curvature, then by [**3**, 4.6] one may split R canonically as $R(\xi, \eta) = W(\xi, \eta) - \{\mathsf{P}(\xi), \eta\} + \{\mathsf{P}(\eta), \xi\}$, where $\mathsf{P} \in \Gamma(T^*M \otimes T^*M)$ is the *rho–tensor* and $W \in \Gamma(\Lambda^2 T^*M \otimes \operatorname{End}_0 TM)$ is called the *Weyl–curvature* of the preferred connection.