Séminaires & Congrès
4, 2000, p. 277–305

ZITTERBEWEGUNG

by

George Sparling

Abstract. — We discuss conjectured relations between twistor theory and superstring theory, built around the idea that time asymmetry is crucial. In the context of a simple example, a number of techniques are described, which should shed light on these conjectures.

***Résumé* (Dynamique du cône de lumière).** — Nous étudions des relations conjecturées entre la théorie des twisteurs et la théorie des supercordes, construites autour de l'idée que l'asymétrie du temps est cruciale. Dans le cadre d'un exemple simple, un certain nombre de techniques sont décrites qui devraient éclaircir ces conjectures.

1. Introduction

The twistor theory associated to flat spacetime may be summarized as follows [**1**–**5**]. First the geometry. We start with a complex vector space $\boldsymbol{T}$, called twistor space, of four complex dimensions, equipped with a pseudo-hermitian sesquilinear form K of signature $(2, 2)$. For $1 \leq n \leq 4$, denote by $\boldsymbol{G}_n$ the Grassmannian of all subspaces of $\boldsymbol{T}$ of dimension n. Then we have a decomposition $\boldsymbol{G}_n = \bigcup_{p+q+r=n} \boldsymbol{G}_{(p,q,r)}$, where for each $V \in \boldsymbol{G}_n$, p, q and r are non-negative integers such that $p + q + r = n$ and $p \leq 2$ is the maximal dimension of a subspace of V on which K is positive definite, whereas $q \leq 2$ is the maximal dimension of a subspace of V on which K is negative definite. Each $\boldsymbol{G}_{(p,q,r)}$ is an orbit of the natural action of the pseudo-unitary group $\boldsymbol{U}(K)$, associated to K, acting on $\boldsymbol{G}_n$ ($\boldsymbol{U}(K)$ is isomorphic to $U(2, 2)$). When $n = 1$, we put $\boldsymbol{PT} = \boldsymbol{G}_1$, $\boldsymbol{PT}^+ = \boldsymbol{G}_{(1,0,0)}$, $\boldsymbol{PT}^- = \boldsymbol{G}_{(0,1,0)}$ and $\boldsymbol{PN} = \boldsymbol{G}_{(0,0,1)}$, so $\boldsymbol{PT} = \boldsymbol{PT}^+ \cup \boldsymbol{PT}^- \cup \boldsymbol{PN}$. $\boldsymbol{PT}$ is a complex projective three-space and $\boldsymbol{PT}^\pm$ are open submanifolds of $\boldsymbol{PT}$, separated by the closed submanifold $\boldsymbol{PN}$, which has real dimension five. In the language of $\boldsymbol{CR}$ geometry, $\boldsymbol{PN}$ is the hyperquadric in

2000 *Mathematics Subject Classification.* — 32G05, 32J81, 32V05.
Key words and phrases. — Time asymmetry, $\boldsymbol{CR}$ geometry, hypersurface twistors, pseudo-Kähler structures.

complex projective three-space, with Levi form of signature $(1,1)$. When $n=2$, the decomposition of the four complex dimensional space $\boldsymbol{CM}=\boldsymbol{G}_2$ has six pieces. Three are open submanifolds: $\boldsymbol{M}^{++}=\boldsymbol{G}_{(2,0,0)}$, $\boldsymbol{M}^{--}=\boldsymbol{G}_{(0,2,0)}$ and $\boldsymbol{M}^{+-}=\boldsymbol{G}_{(1,1,0)}$. Another is a closed subset, of real dimension four: $\boldsymbol{M}=\boldsymbol{G}_{(0,0,2)}$. The other two, $\boldsymbol{M}^{+}=\boldsymbol{G}_{(1,0,0)}$ and $\boldsymbol{M}^{-}=\boldsymbol{G}_{(0,1,0)}$ each have real dimension seven and have $\boldsymbol{M}$ as their boundary. The boundary of $\boldsymbol{M}^{++}$ is $\boldsymbol{M}^{+}\cup\boldsymbol{M}$, of $\boldsymbol{M}^{--}$ is $\boldsymbol{M}^{-}\cup\boldsymbol{M}$ and of $\boldsymbol{M}^{+-}$ is $\boldsymbol{M}^{+}\cup\boldsymbol{M}^{-}\cup\boldsymbol{M}$. Each point of $\boldsymbol{CM}$ is a projective line in $\boldsymbol{PT}$. Then $\boldsymbol{M}^{++}$, $\boldsymbol{M}^{--}$, $\boldsymbol{M}^{+-}$, $\boldsymbol{M}^{\pm}$ and $\boldsymbol{M}$ are, respectively, the spaces of projective lines that lie entirely in $\boldsymbol{PT}^{+}$, lie entirely in $\boldsymbol{PT}^{-}$, cross from $\boldsymbol{PT}^{+}$ to $\boldsymbol{PT}^{-}$, touch $\boldsymbol{PN}$ at one point and otherwise lie in $\boldsymbol{PT}^{\pm}$, lie entirely inside $\boldsymbol{PN}$. $\boldsymbol{G}_3$ is isomorphic to dual projective twistor space, the projective dual of $\boldsymbol{PT}$ and has three pieces, $\boldsymbol{G}_{(2,1,0)}$, $\boldsymbol{G}_{(1,2,0)}$ and $\boldsymbol{G}_{(1,1,1)}$.

The Klein correspondence embeds $\boldsymbol{CM}$ as a quadric hypersurface in the projective space of $\Omega^2\boldsymbol{T}$, the exterior product of $\boldsymbol{T}$ with itself. As such it inherits a natural conformally flat complex holomorphic conformal structure. Two points of $\boldsymbol{CM}$ are null related if and only if their corresponding lines in $\boldsymbol{PT}$ intersect. Then $\boldsymbol{CM}$ is the complexification of $\boldsymbol{M}$ and the conformal structure of $\boldsymbol{M}$ is real and Lorentzian. $\boldsymbol{M}$ is a conformal compactification of real Minkowski spacetime. If a specific point $\boldsymbol{I}$ of $\boldsymbol{M}$, is singled out, then on the complement $\boldsymbol{M}_{\boldsymbol{I}}$ of the null cone of $\boldsymbol{I}$, $\boldsymbol{M}$ has a canonical flat Lorentzian metric, and $\boldsymbol{M}_{\boldsymbol{I}}$ (of topology $\boldsymbol{R}^4$) may be regarded as Minkowski spacetime.

With respect to the real Minkowski space $\boldsymbol{M}_{\boldsymbol{I}}$, the imaginary part y, of the position vector of a finite point of $\boldsymbol{CM}$, is canonical. Then $\boldsymbol{M}^{++}$, $\boldsymbol{M}^{--}$, $\boldsymbol{M}^{+-}$, $\boldsymbol{M}^{+}$ and $\boldsymbol{M}^{-}$ are the sets of all points of $\boldsymbol{CM}$, for which y is respectively, past pointing and timelike, future pointing and timelike, spacelike, past pointing and null, future pointing and null.

Each point of $\boldsymbol{PT}$ (called a projective twistor) may be represented as a completely null two-surface in $\boldsymbol{CM}$. This surface intersects $\boldsymbol{M}$, if and only if the projective twistor lies in $\boldsymbol{PN}$ and then the intersection is a null geodesic. The induced mapping from $\boldsymbol{PN}$ to the space of null geodesics in $\boldsymbol{M}$ turns out to be a natural isomorphism, yielding the key fact that the space of null geodesics in $\boldsymbol{M}$ is naturally a $\boldsymbol{CR}$ manifold, such that there is a one-to-one correspondence between points of $\boldsymbol{M}$ and Riemann spheres embeddded in the $\boldsymbol{CR}$ manifold. The null cone of $\boldsymbol{I}$, called scri, is an asymptotic null hypersurface for the Minkowski spacetime. There are now three different kinds of null cones: the null cone of a finite point (a point of $\boldsymbol{M}_{\boldsymbol{I}}$), scri itself, which has no finite points and the null cone of a point of scri, distinct from $\boldsymbol{I}$. This latter kind of cone intersects scri in a null geodesic and intersects $\boldsymbol{M}_{\boldsymbol{I}}$ in a null hyperplane.

Analytically, we find that the information in solutions of certain relativistic field equations on $\boldsymbol{M}$ or on $\boldsymbol{CM}$ is encoded in global structure in $\boldsymbol{PT}$: for example, the first sheaf cohomology group of suitable domains in $\boldsymbol{PT}$ with coefficients in the sheaf of germs of holomorphic functions on $\boldsymbol{PT}$ corresponds to the space of solutions of the anti-self-dual Maxwell equations on the corresponding domain in $\boldsymbol{CM}$. In particular for the domains $\boldsymbol{PT}^+$ and $\boldsymbol{PT}^-$, the solutions are global on $\boldsymbol{M}^{++}$ and $\boldsymbol{M}^{--}$ respectively. For solutions in $\boldsymbol{M}$ only we use instead $\boldsymbol{CR}$ cohomology on subsets of $\boldsymbol{PN}$. This has the key advantage that non-analytic solutions are encompassed. If we pass to suitable vector bundles over $\boldsymbol{PT}$, or over $\boldsymbol{PN}$, then we encode the information of solutions of the anti-self-dual Yang-Mills equations. Also each holomorphic surface in $\boldsymbol{PT}$ intersects $\boldsymbol{PN}$ in a three-space. This space gives rise to a shear-free null congruence in Minkowski spacetime and all analytic shear free congruences are obtained this way. Non-analytic shear-free null congruences can be constructed. In general, they appear to be represented by holomorphic surfaces in either $\boldsymbol{PT}^+$, or $\boldsymbol{PT}^-$, that extend to the boundary $\boldsymbol{PN}$, but no further: such surfaces are said to be one-sided embeddable.

Given this elegant theory for flat spacetime, it is natural to ask to extend the theory to curved spacetime. Here a fundamental obstacle immediately arises, even for real analytic spacetimes. The twistors in flat spacetime are interpreted as completely null two-surfaces and it is easy to prove that such surfaces can exist, in the required generality, if and only if the spacetime is conformally flat. In the language of the Frobenius theorem, the twistor surfaces are described by a system of one-forms and the integrability of the system forces conformal flatness. Penrose realized that if the dimension was reduced by one, then the integrability problem would be overcome and a twistor theory could then be constructed [**3**]. Specifically, the curved analogue of the twistor distribution is integrable when restricted to the spin bundle over a hypersurface in spacetime, so each hypersurface in spacetime has an associated twistor theory.

If the spacetime is asympotically flat, then there are attached to the spacetime, two asymptotic null cones, one in the future and one in the past, called scri plus and scri minus, respectively. Newman and Penrose were able to completely analyze the twistor structures of these spaces, called $\boldsymbol{H}$-spaces [**6, 7, 8, 10, 12**]. Each projective twistor is represented in the surface by an appropriate complex null geodesic curve (if p^a is tangent to the curve and if n^a is the normal to the surface, then necessarily the outer product $p^{[a}n^{b]}$ is either self-dual or anti-self-dual; for twistors this outer product must be anti-self-dual; the self-dual alternative gives the "dual" or "conjugate" twistor space; the information in each space is the same). Then the space of such curves is three complex dimensional, as in the flat case. The space is fibered over a complex projective one-space (a Riemann sphere) and in favorable circumstances, there is a four complex parameter set of sections of the fibering (so each section is a Riemann sphere embedded in the projective twistor space) [**21**]. This gives a curved analogue of the space $\boldsymbol{CM}$ of flat twistor space. Just as for flat space, a complex conformal structure is determined by the incidence condition for the holomorphic sections and a preferred holomorphic metric may be defined in this conformal class. This metric is then shown to be vacuum and to have anti-self-dual Weyl curvature. Finally there is a non-projective twistor space obtained by propagating a spinor along the projective twistor curve and this non-projective space has a pseudo-Kähler structure, K, whose associated metric is Ricci flat. We then have curved analogues of some of the various spaces $\boldsymbol{G}_{(p,q,r)}$ discussed above. In particular, the vanishing of K determines a $\boldsymbol{CR}$ hypersurface in the twistor space, which, in turn, may be interpeted as the bundle of null directions over the asymptotic null hypersurface of the spacetime.

The success of the asymptotic twistor theory of Newman and Penrose raises the question of extending the theory to the finite realm. Here one notes that the asymptotic twistor theory is still rather special in that first, scri is a null hypersurface and secondly, that it is shearfree. For a null hypersurface the hypersurface twistor curves are complex null geodesics in the surface, if and only if the surface is shearfree. Geometrically, shearfreeness amounts to the fact that the complexification of scri is foliated by a one complex parameter set of completely null two-surfaces, which cannot exist away from infinity except for certain hypersurfaces in algebraically special spacetimes. Nevertheless one might anticipate that some sort of deformation of the Newman-Penrose theory is required. Indeed, for twistor spaces associated to spacelike hypersurfaces, this is the case, if analyticity is assumed [**9**].

In recent seminars, I have suggested that the Newman-Penrose picture breaks down, at least, for the properly constructed twistor spaces of finite null cones [**17**–**20**], the mechanism for the breakdown being provided by the Sachs equations [**32**]. These ideas are detailed in the appendix here. Instead I suggest that the twistor spaces of these null cones will be complex manifolds more like those that appear in string theory and that these twistor spaces will then provide a link between the string theory and spacetime theory. Specifically in string theory, complex manifolds with isolated compact Riemann spheres (or surfaces of higher genus) play an important role. Essentially, I am saying that the spheres of string theory are to be identified conceptually and theoretically with isolated spheres in the null hypersurface twistor spaces. String theorists assert that their theory incorporates gravity. To the limited extent that I understand their theory, I would respond that they may well have gravitational degrees of freedom in the theory, in the sense for example that they consistently construct models of gravitating particles, but they do not yet incorporate all the subtleties of the Einstein theory and that it may be that a more complete theory will require a unification of string-theoretic, twistor-theoretic and other ideas. *In the new theory, time asymmetry would be natural.* Also even "local" physics would depend via the structure of null cone hypersurface twistor spaces on the global past of the locality. This would apparently mean that there would be very subtle deviations from $\boldsymbol{PCT}$ invariance in local physics, the main point here being that the global structure of past null cones differs from that of future null cones.

In trying to analyze whether or not these conjectures are in any way sensible, we should be careful to frame the discussion properly. Also we should realize that we are in a no-lose situation. Any progress in this analysis, whether positive or negative, relative to these conjectures, will result in substantial gains in knowledge. Certainly global questions come into play; for example $\boldsymbol{C}^3$ and complex projective three-space differ only at "infinity", but the former has no embedded Riemann spheres, whilst the latter has a four-parameter set. Also, as in flat space, there are many kinds of null cone hypersurfaces; the theory of each kind will have its own flavor. The list includes the past and future null cones of a finite point; null cones avoiding singularities, null cones of points in or on horizons; cosmological null cones; "virtual" null cones: scri plus, scri minus, horizons, null cones of singular points, of points of scri, of points beyond scri. Unfortunately, when trying to construct examples, one is practically forced to use analytic spacetimes, whereas the key to the Einstein theory is its hyperbolic nature, which truly can be exposed only in a non-analytic framework. So one must instead adopt the following philosphical schema: when working with analytic spacetimes, avoid any construction that has no hope of a non-analytic analogue; also avoid bringing in any information which in a non-analytic situation would violate causality. In particular, this entails that we should emphasize the role of the $\boldsymbol{CR}$ twistor manifolds at every opportunity.

The present work gives the first example of the twistor theory of null hypersurfaces, for the case of a shearing null hypersurface. Even in the very simple case, discussed here, the computations are somewhat non-trivial and at various steps were aided by the Maple algebraic computing system. The title of this work refers to the idea prevalent in quantum field theory that dynamics proceeds along the null cone, progress in a timelike direction being made as a zigzag along various null cones, alternately future and past pointing. If my conjectures have any sense, the analogous idea in string theory is chains or ensembles of manifolds of Calabi-Yau type, connected by webs of mirror symmetries. Here I confine myself to working out some of the relevant formulas of the twistor theory. In particular an example of twistor scattering is constructed, I believe for the first time in the literature. The scattering in question depends essentially on the spacetime not being conformally flat. Two null cones intersect in a two-surface. A twistor curve of one cone meets the two-surface at one point. The attached spinor to the curve then naturally gives rise to a new twistor curve on the second null cone. This gives rise to a local diffeomorphism between the two twistor spaces, this diffeomorphism being the Zitterbewegung.

It seems possible, although I do not yet have a proof, that this scattering will be feasible even in the non-analytic case, at least for suitable spacetimes and thus be consistent with my overall philosophy. This would entail that the twistor $\boldsymbol{CR}$