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HOLONOMIC AND SEMI-HOLONOMIC GEOMETRIES

by

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Abstract. — Holonomic and semi-holonomic geometries modelled on a homogeneous space G/P are introduced as reductions of the holonomic or semi-holonomic frame bundles respectively satisfying a straightforward generalization of the partial differential equation characterizing torsion–free linear connections. Under a suitable regularity assumption on the model space G/P we establish an equivalence of categories between Cartan geometries and semi-holonomic geometries modelled on G/P.

***Résumé* (Géométries holonomes et semi–holonomes).** — On introduit les géométries holonomes et semi–holonomes modelées sur un espace homogène G/P comme réductions des fibrés de repères holonomes et semi–holonomes vérifiant une généralisation de l'équation aux dérivées partielles caractérisant les connexions linéaires sans torsion. Sous certaines conditions de régularité sur l'espace modèle G/P, nous établissons une équivalence de catégories entre les géométries de Cartan et les géométries semi–holonomes modelées sur G/P.

1. Introduction

The study of geometric structures with finite dimensional isometry groups has ever made up an important part of differential geometry and is intimately related with the notions of connections and principal bundles, coined by Cartan in order to give an interpretation of Lie's ideas on geometry. Principal bundles are undoubtedly useful in the study of geometric structures on manifolds, nevertheless one should not fail to notice the problematic and somewhat paradox aspect of their use. In fact the frame bundles of a manifold M are defined as jet bundles, with a single projection to M, say the target projection, but we have to keep track of the source projection, too. From the point of view of exterior calculus on principal bundles there is a natural way to work around this problem, needless to say it was Cartan who first treated the classical examples of geometric structures along these lines of thought, which have by now become standard. The paradox itself however remains and its impact is easily

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noticed when turning to more general geometric structures, say geometries modelled on homogeneous spaces G/P.

Analysis on homogeneous spaces G/P is well understood and it is tempting to generalize this analysis to curved analogues of the flat model space G/P. In particular the extension problem for invariant differential operators studied in conformal and more general parabolic geometries only makes sense in this context. Cartan's original definition [**C**] of Cartan geometries as curved analogues of homogeneous spaces G/P relies on the existence of an auxiliary principal bundle $\mathcal{G}$ on a manifold M. Unless we are content with studying pure Cartan geometries we need to discover the geometry first in order to establish the existence of the principal bundle. In fact most Cartan geometries arise via Cartan's method of equivalence in the process of classifying underlying geometric structures interesting in their own right. In this respect the work of Tanaka [**T**] has been most influential, who introduced parabolic Cartan geometries to classify regular differential systems with simple automorphism groups.

An alternative, but essentially equivalent definition of a curved analogue of a homogeneous space is introduced in this note. Holonomic and semi-holonomic geometries modelled on a homogeneous space G/P will be reductions of the holonomic or semi-holonomic frame bundles $\mathbf{GL}^{d}M$ or $\overline{\mathbf{GL}}^{d}M$ of M satisfying a suitable partial differential equation, which is a straightforward generalization of the partial differential equation characterizing torsion–free linear connections as reductions of $\mathbf{GL}^{2}M$ to the structure group $\mathbf{GL}^{1}\mathbb{R}^n \subset \mathbf{GL}^{2}\mathbb{R}^n$. The critical step in the formulation of this partial differential equation is the construction of a map similar to

$$\mathcal{J}: \quad \mathbf{O}\,\mathbb{R}^n\backslash\mathbf{GL}^{2}\mathbb{R}^n \;\longrightarrow\; \mathrm{Jet}^1_0(\mathbf{O}\,\mathbb{R}^n\backslash\mathbf{GL}^{1}\mathbb{R}^n)$$

in Riemannian and

$$\mathcal{J}: \quad \mathbf{CO}\,\mathbb{R}^n \ltimes \mathbb{R}^{n*}\backslash\mathbf{GL}^{2}\mathbb{R}^n \;\longrightarrow\; \mathrm{Jet}^1_0(\mathbf{CO}\,\mathbb{R}^n\backslash\mathbf{GL}^{1}\mathbb{R}^n)$$

in conformal geometry. The classical construction of $\mathcal{J}$ applies only for affine geometries, i. e. geometries modelled on quotients of the form $P \ltimes \mathfrak{u}/P$, where the semidirect product is given by some linear representation of P on $\mathfrak{u}$. In non–affine geometries the straightforward map $\mathbf{GL}^{d+1}\mathbb{R}^n \longrightarrow \mathrm{Jet}^1_0\mathbf{GL}^{d}\mathbb{R}^n$ fails in general to descend to quotients. In particular this problem arises in split geometries, which are of particular interest in differential geometry. Split geometries are modelled on homogeneous spaces G/P, such that some subgroup $U \subset G$ acts simply transitively on an open, dense subset of G/P. A couple of talks at the conference in Luminy centered about parabolic geometries, which form a class of examples of split geometries interesting in its own right due to the existence of the Bernstein–Gelfand–Gelfand resolution [**BE**], [**CSS**].

Without loss of generality we will assume that the model space G/P is connected, i. e. every connected component of G meets P. However G/P will have to satisfy a technical regularity assumption in order to be able to construct holonomic and semi-holonomic geometries modelled on G/P. Choose a linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and consider the corresponding exponential coordinates of G/P:

$$\exp: \quad \mathfrak{u} \;\longrightarrow\; G/P, \quad v \;\longmapsto\; e^{v}P$$

The action of the isotropy group P of $\exp(0) = eP$ in these exponential coordinates gives rise to a group homomorphism $\Phi_{\mathfrak{u}} : P \longrightarrow \mathbf{GL}^k\mathfrak{u}$ from P to the group $\mathbf{GL}^k\mathfrak{u}$ of k–th order jets of diffeomorphisms of $\mathfrak{u}$ into itself fixing $0 \in \mathfrak{u}$. We require that the image of P is closed in $\mathbf{GL}^k\mathfrak{u}$ for all $k \geq 1$, a condition evidently independent of the choice of $\mathfrak{u}$. This regularity assumption is certainly met by all pairs of algebraic groups, but it does not hold in general, perhaps the simplest counterexample is the affine geometry modelled on $\mathbb{R} \ltimes (\mathbb{C} \oplus \mathbb{C})/\mathbb{R}$ with $\mathbb{R}$ acting on $\mathbb{C} \oplus \mathbb{C}$ by an irrational line in $S^1 \times S^1$. In general neither of the homomorphisms $P \longrightarrow \mathbf{GL}^k\mathfrak{u}$, $k \geq 1$, needs be injective, however the intersection of all their kernels is a closed normal subgroup P_∞ of P called the isospin group of P in G. Alternatively P_∞ can be characterized as the kernel of the homomorphism $G \longrightarrow \operatorname{Diff} G/P$.

In the absence of isospin $P_\infty = \{1\}$ Morimoto [**M**] constructed a P–equivariant embedding of a Cartan geometry $\mathcal{G}$ on a manifold M into the infinite frame bundle $\mathcal{G} \longrightarrow \overline{\mathbf{GL}}^\infty M$. The main result of the current note is a generalization of this result, which provides a complete classification of Cartan geometries $\mathcal{G}$ on M modelled on G/P in terms of semi-holonomic geometries of sufficiently high order:

Theorem 1.1. — *Consider a connected homogeneous quotient G/P of a finite dimensional Lie group G by a closed subgroup P such that the image of P in $\mathbf{GL}^k\mathfrak{u}$ is closed for all $k \geq 1$. There exists an integer $d \geq 0$ depending only on the pair of Lie algebras $\mathfrak{g} \supset \mathfrak{p}$ such that every Cartan geometry $\mathcal{G}$ on M is an isospin P_∞–bundle over a unique semi-holonomic geometry $\mathcal{G}/P_\infty \subset \overline{GL}^{d+1}M$ of order $d+1$ modelled on G/P. The semi-holonomic geometry fixes the Cartan connection on $\mathcal{G}$ up to an affine subspace of isospin connections.*

Consequently in the absence of isospin $P_\infty = \{1\}$ there is a natural correspondence between Cartan geometries and semi-holonomic geometries of order $d+1$ on M establishing an equivalence of the respective categories. The actual proof of Theorem 1.1 is very simple once we forget everything we learned about the canonical connection etc. on frame bundles. The explanation for the need to introduce an auxiliary bundle in the original definition of Cartan geometries seems to be that people clinged to the concept of "canonical" translations, because it fitted so neatly with exterior calculus, instead of taking the problematic aspect of principal bundles in geometry at face value.

It is a striking fact that no classical example is known where the integer d in Theorem 1.1 is different from $d = 1$ or $d = 2$. In fact the relationship between Cartan geometries and holonomic geometries should become very interesting for examples with $d > 2$. A partial negative result in this direction is given in Lemma 4.4 showing that all examples with reductive G have $d \leq 2$.

Perhaps the most important aspect of Theorem 1.1 is that it associates a classifying geometric object and thus local covariants to any Cartan geometry without any artificial assumptions on the model space G/P. In particular the techniques available in the formal theory of partial differential equations or exterior differential systems [$\mathbf{BCG}^3$] can be used to describe the space of local solutions to the partial differential equation characterizing holonomic and semi-holonomic geometries. The most ambitious program is to derive the complete resolution of the space of local covariants and we hope to return to this project in [**W**]. The methods and results of Tanaka [**T**] and Yamaguchi [**Y**] for parabolic geometries will certainly find their place in the more general context of split geometries.

In the following section we will review the fundamentals of jet theory with particular emphasis on the delicate role played by the translations in order to construct the map $\mathcal{J}$ for all model spaces G/P. Moreover we will review the notion of torsion in this section, because similar to the map $\mathcal{J}$ the most intuitive definition of torsion depends on the choice of translations. This example is particularly interesting, because it contradicts the usual definition of torsion as the exterior derivative of the soldering form and may serve as a sample calculation showing the way the translations affect the relevant formulas in exterior calculus.

Using the map $\mathcal{J}$ we set up the partial differential equation characterizing holonomic and semi-holonomic reductions of the holonomic and semi-holonomic frame bundles $\mathbf{GL}^{d}M$ and $\overline{\mathbf{GL}}^{d}M$ respectively. In particular we will provide stable versions of these partial differential equations, a problem we thought about at the time of the conference in Luminy. Moreover we will discuss what kind of connections are associated with holonomic and semi-holonomic reductions. In the final section we prove Theorem 1.1 and thus establish an equivalence of categories between the category of Cartan geometries and the category of semi-holonomic geometries of sufficiently high order.

I would like to thank the organizers of the conference for inviting me to Luminy and giving me extra time to finish this note. Moreover the discussions with Jan Slovák and Lukáš Krump in Luminy turned my attention to the local covariant problem in pure Cartan geometry. My special thanks are due to Tammo Diemer, who introduced me to conformal geometry and the related extension problem for invariant differential operators.

2. Jet Theory and Principal Bundles

The language of jet theory will dominate the following sections, most of the ideas and definitions will emerge from this way of expressing calculus. Since there are numerous text books on this subject it is needless to strive for a detailed introduction, see [**KMS**], [**P**] for further reference. For the convenience of the reader we want to recall the basic concepts and definitions of jet theory and discuss its interplay with the theory of principal bundles. In particular we want to point out the problematic aspect of using principal bundles in the description of jets of geometric structures on manifolds. In order to get a well defined projection from a principal bundle to the base manifold we have to fix say the target of a jet, however we have to keep track of its source, too.

Perhaps the cleanest way around this problem is to discard principal bundles and turn to groupoid–like structures. In fact the description of geometric structures on manifolds using groupoids or better Lie pseudogroups has a long history originating from Lie and predating the concept of principal bundles by decades, see [**P**] for an enthusiastic and in parts rather polemical historical survey. On the other hand the use of principal bundles has a tremendous advantage over the use of groupoids, we really can do calculations without the need to resort to local coordinates and the powerful algebraic machinery of resolutions by induced modules becomes available in this context.

There is a standard recipe to deal with this dichotomy and it works remarkably well in affine and other important geometries. Moreover it links neatly with exterior calculus on principal bundles pioneered by Cartan. In this note we will explore variants of the standard recipe depending in geometrical language on the choice of translations. Although these variants may look somewhat artificial from the point of view of exterior calculus they allow us to deal easily not only with affine but with all split geometries. A striking example is Lemma 2.5, which essentially reproduces the definition of torsion in Cartan geometries without any reference to connections at all. The modifications in the definitions needed in general geometries modelled on homogeneous spaces G/P will appear in [**W**].

The main object of study in jet theory is of course a jet, which is a generalization of the concept of a Taylor series associated to a smooth map $\mathbb{R} \longrightarrow \mathbb{R}$ to arbitrary smooth maps between manifolds. Let $\mathfrak{u}$ be a fixed real vector space and $\mathcal{F}$ some differentiable manifold. Two smooth maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ and $\tilde{f} : \mathfrak{u} \longrightarrow \mathcal{F}$ defined in some neighborhood of $0 \in \mathfrak{u}$ are called equivalent $f \sim \tilde{f}$ up to order $k \geq 0$ if $f(0) = \tilde{f}(0)$ and their partial derivatives up to order k in some and hence every local coordinate system of $\mathcal{F}$ about $f(0) = \tilde{f}(0)$ agree in 0. The equivalence class of a smooth map f up to order k is called the k–th order jet $\mathrm{jet}_0^k f$ of f and the set of all these equivalence classes is denoted by $\mathrm{Jet}_0^k \mathcal{F} := \{\mathrm{jet}_0^k f |\ f : \mathfrak{u} \longrightarrow \mathcal{F}\ \}$. For all $k \geq l \geq 0$ there is a canonical projection

$$\mathrm{pr} : \quad \mathrm{Jet}_0^k \mathcal{F} \longrightarrow \mathrm{Jet}_0^l \mathcal{F}, \quad \mathrm{jet}_0^k f \longmapsto \mathrm{jet}_0^l f$$

and the evaluation

$$\mathrm{ev} : \quad \mathrm{Jet}_0^k \mathcal{F} \longrightarrow \mathcal{F}, \quad \mathrm{jet}_0^k f \longmapsto f(0)$$

which strictly speaking is a special case of the projection since we may identify $\mathrm{Jet}_0^0 \mathcal{F} \cong \mathcal{F}$. We will use a different notation for this special case nevertheless to avoid the cumbersome indication of the source and target orders of the projections. If the manifold $\mathcal{F}$ comes along with a distinguished base point $\{*\}$ the jets of pointed smooth maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ make up the subset of all reduced or pointed jets ${}^*\mathrm{Jet}_0^k \mathcal{F} = \{\mathrm{jet}_0^k f |\ f(0) = *\} \subset \mathrm{Jet}_0^k \mathcal{F}$, which is just the preimage $\mathrm{ev}^{-1}(*) = {}^*\mathrm{Jet}_0^k \mathcal{F}$ of the base point.

Consider now the case that Q is a Lie group then so are both ${}^*\mathrm{Jet}_0^k Q$ and $\mathrm{Jet}_0^k Q$ under pointwise multiplication with Lie algebras ${}^*\mathrm{Jet}_0^k \mathfrak{q}$ and $\mathrm{Jet}_0^k \mathfrak{q}$ respectively. With the help of the exponential $\exp : \mathfrak{q} \longrightarrow Q$ we may identify ${}^*\mathrm{Jet}_0^k Q$ and ${}^*\mathrm{Jet}_0^k \mathfrak{q}$, making the vector space ${}^*\mathrm{Jet}_0^k \mathfrak{q}$ an algebraic group with group structure given by the polynomial approximation of the Campbell–Baker–Hausdorff formula. The group