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PSEUDO-RIEMANNIAN METRICS WITH PARALLEL SPINOR FIELDS AND VANISHING RICCI TENSOR

by

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Abstract. — I will discuss geometry and normal forms for pseudo-Riemannian metrics with parallel spinor fields in some interesting dimensions. I also discuss the interaction of these conditions for parallel spinor fields with the Einstein equations.

***Résumé* (Métriques pseudo-riemanniennes admettant des spineurs parallèles et un tenseur de Ricci nul)**

Je discuterai la géométrie et les formes normales pour les métriques pseudo-riemanniennes qui ont des champs de spineurs parallèles en quelques dimensions intéressantes. Je discuterai aussi l'interaction de ces conditions pour les champs de spineurs parallèles avec les équations d'Einstein.

1. Introduction

1.1. Riemannian holonomy and parallel spinors. — The possible restricted holonomy groups of irreducible Riemannian manifolds have been known for some time now [**2, 6, 7**]. The list of holonomy-irreducible types in dimension n that have nonzero parallel spinor fields is quite short: The holonomy H of such a metric must be one of

- $H = \mathrm{SU}(m)$ (i.e., special Kähler metrics in dimension $n = 2m$);
- $H = \mathrm{Sp}(m)$ (i.e., hyper-Kähler metrics in dimensions $n = 4m$);
- $H = \mathrm{G}_2$ (when $n = 7$); or
- $H = \mathrm{Spin}(7)$ (when $n = 8$).

In Cartan's sense, the local generality [**6, 7**] of metrics with holonomy

- $H = \mathrm{SU}(m)$ ($n = 2m$) is 2 functions of $2m{-}1$ variables,
- $H = \mathrm{Sp}(m)$ ($n = 4m$) is $2m$ functions of $2m{+}1$ variables,
- $H = \mathrm{G}_2$ ($n = 7$) is 6 functions of 6 variables, and
- $H = \mathrm{Spin}(7)$ ($n = 8$) is 12 functions of 7 variables.

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In each case, a metric with holonomy H has vanishing Ricci tensor.

1.2. Relations with physics. — The existence of parallel spinor fields seems to account for much of the interest in metrics with special holonomy in mathematical physics, since such spinor fields play a central role in supersymmetry. In the case of string theory, SU(3), and lately, with the advent of $\mathcal{M}$-theory, G_2 (and possibly even Spin(7)) seem to be of interest. I don't know much about these physical theories, so I will not attempt to discuss them.

1.3. Pseudo-Riemannian generalizations. — In the past few years, I have been asked by a number of physicists about the generality of pseudo-Riemannian metrics satisfying conditions having to do with parallel spinors and with solutions of the Einstein equations. (In contrast to the Riemannian case, an indecomposable pseudo-Riemannian metric can possess a parallel spinor field without being Einstein.)

For example, there seems to be some current interest in Lorentzian manifolds of type $(10, 1)$ having parallel spinor fields and perhaps also having vanishing Ricci curvature, about which I will have more to say later in the article.

Recall [**17, 5**] that in the pseudo-Riemannian case, there is a distinction to be made between a metric being holonomy-irreducible (no parallel subbundles of the tangent bundle), being holonomy-indecomposable (no parallel splitting of the tangent bundle), and being indecomposable (no local product decomposition of the metric). (In the Riemannian case, of course, these conditions are locally equivalent.) The classification of the holonomy-irreducible case proceeds much as in the positive definite case [**8**], but an indecomposable pseudo-Riemannian metric need not be holonomy irreducible. It is this difference that makes classifying the possible pseudo-Riemannian metrics having parallel spinor fields something of a challenge. For a general discussion of the differences, particularly the failure of the de Rham splitting theorem, see [**3, 4**]. Also, the results and examples in [**13, 14**] are particularly illuminating.

Now, quite a lot is known about the pseudo-Riemannian case when the holonomy acts irreducibly. For a general survey in this case, particularly regarding the existence of parallel spinor fields, see [**1**]. Note that, in all of these cases, the Ricci tensor vanishes. This is not so when the holonomy acts reducibly. Already in dimension 3, Lorentzian metrics can have parallel spinor fields without being Ricci-flat.

An intriguing relationship between the condition for having a parallel spinor and the Ricci equations came to my attention after a discussion during a 1997 summer conference in Edinburgh with Ines Kath. It had been known for a while [**6**] that the metrics in dimension 7 with holonomy G_2 depend locally on six functions of six variables (modulo diffeomorphism). Now, the condition of having holonomy in G_2 is equivalent to the condition of having a parallel spinor field. I had also shown that the $(4, 3)$-metrics with holonomy G_2^* depend locally on six functions of six variables, and the condition of having this holonomy in this group is the same as the condition that the $(4,3)$-metric admit a non-null parallel spinor field. Ines Kath had noticed that the structure equations of a $(4,3)$ metric with a null parallel spinor field did not seem to imply that the Ricci curvature vanished, and she wondered whether or not there existed examples in which it did not. After some analysis, I was able to show that there are indeed $(4,3)$-metrics with parallel spinor fields whose Ricci curvature is not zero and whose holonomy is equal to the full stabilizer of a null spinor. These metrics depend on three arbitrary functions of seven variables. However, a more intriguing result is that, when one combines the condition of having a parallel null spinor with the condition of being Ricci-flat, the $(4,3)$-metrics with this property depend on six functions of six variables, just as in the non-null case (where the vanishing of the Ricci tensor is automatic).

In any case, this and the questions from physicists motivates the general problem of determining the local generality of pseudo-Riemannian metrics with parallel spinors, with and without imposing the Ricci-flat condition. This article will attempt to describe some of what is known and give some new results, particularly in dimensions greater than 6.

Most of the normal forms that I describe for metrics with parallel spinor fields of various different algebraic types are already known in the literature, or have been derived independently by others. (In particular, Kath [**15**] has independently derived the normal forms for the split cases with a pure parallel spinor.) What I find the most interesting is that, in every known case, the system of PDE given by the Ricci-flat condition is either in involution (in Cartan's sense) with the system of PDE that describe the (p,q)-metrics with a parallel spinor of given algebraic type or else follows as a consequence (and so, in a manner of speaking, is trivially in involution with the parallel spinor field condition). I have no general proof that this is so in all cases, nor even a precise statement as to how general the solutions should be, since this seems to depend somewhat on the algebraic type of the parallel spinor. What does seem to be true in a large number of (though not all) cases, though, is that the local generality of the Ricci-flat (p,q)-metrics with a parallel spinor of a given algebraic type seems to be largely independent of the given algebraic type, echoing the situation for $(4,3)$-metrics mentioned above that first exhibited this phenomenon.

Since this article is mainly a discussion of cases, together with an explicit working out of the standard moving frame methods and applications of Cartan-Kähler theory, I cannot claim a great deal of originality for the results. Consequently, I do not state the results in the form of theorems, lemmas, and propositions, but instead discuss each case in turn. The most significant results are probably the descriptions of the generality of the Ricci-flat metrics with parallel spinors in the various cases. Another possibly significant result is the description of the $(10,1)$-metrics with a parallel null spinor field, since this seems to be of interest in physics [**11**].

2. Algebraic background on spinors

All of the material in this section is classical. I include it to fix notation and for the sake of easy reference for the next section. For more detail, the reader can consult [**12, 16**].

2.1. Notation. — The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ denote, as usual, the rings of real numbers, complex numbers, quaternions, and octonions, respectively. When $\mathbb{F}$ is one of these rings, the notation $\mathbb{F}(n)$ means the ring of n-by-n matrices with entries in $\mathbb{F}$. The notation $\mathbb{F}^n$ will always denote the space of *column* vectors of height n with entries in $\mathbb{F}$. Vector spaces over $\mathbb{H}$ will always be regarded as having the scalar multiplication acting on the *right*. For an m-by-n matrix a with entries in $\mathbb{C}$ or $\mathbb{H}$, the notation a^* will denote its conjugate transpose. When a has entries in $\mathbb{R}$, a^* will simply denote the transpose of a.

The notation $\mathbb{R}^{p,q}$ denotes $\mathbb{R}^{p+q}$ endowed with an inner product of type (p,q). The notation $\mathbb{C}^{p,q}$ denotes $\mathbb{C}^{p+q}$ endowed with an Hermitian inner product of type (p,q), with a similar interpretation of $\mathbb{H}^{p,q}$, but the reader should keep in mind that a quaternion Hermitian inner product satisfies $\langle v, wq\rangle = \langle v,w\rangle q$ and $\langle vq, w\rangle = \bar{q}\langle v,w\rangle$ for $q \in \mathbb{H}$.

2.2. Clifford algebras. — The Clifford algebra $\mathrm{C}\ell(p,q)$ is the associative algebra generated by the elements of $\mathbb{R}^{p,q}$ subject to the relations $vw+wv = -2v{\cdot}w\,\mathbf{1}$. This is a $\mathbb{Z}_2$-graded algebra, with the even subalgebra $\mathrm{C}\ell^e(p,q)$ generated by the products vw for $v, w \in \mathbb{R}^{p,q}$.

Because of the following formulae, valid for $p, q \geq 0$ (see [**12, 16**]),

$$
\begin{aligned}
\mathrm{C}\ell^e(p{+}1,q) &\simeq \mathrm{C}\ell(p,q)\\
\mathrm{C}\ell(p{+}1,q{+}1) &\simeq \mathrm{C}\ell(p,q)\otimes\mathrm{C}\ell(1,1)\\
\mathrm{C}\ell(p{+}8,q) &\simeq \mathrm{C}\ell(p,q)\otimes\mathrm{C}\ell(8,0)\\
\mathrm{C}\ell(p,q{+}1) &\simeq \mathrm{C}\ell(q,p{+}1)
\end{aligned}
\tag{1}
$$

all these algebras can be worked out from the table

$$
\begin{aligned}
&\mathrm{C}\ell(0,1) \simeq \mathbb{R}\oplus\mathbb{R} && \mathrm{C}\ell(1,1) \simeq \mathbb{R}(2)\\
&\mathrm{C}\ell(1,0) \simeq \mathbb{C} && \mathrm{C}\ell(2,0) \simeq \mathbb{H}\\
&\mathrm{C}\ell(3,0) \simeq \mathbb{H}\oplus\mathbb{H} && \mathrm{C}\ell(4,0) \simeq \mathbb{H}(2)\\
&\mathrm{C}\ell(5,0) \simeq \mathbb{C}(4) && \mathrm{C}\ell(6,0) \simeq \mathbb{R}(8)\\
&\mathrm{C}\ell(7,0) \simeq \mathbb{R}(8)\oplus\mathbb{R}(8) && \mathrm{C}\ell(8,0) \simeq \mathbb{R}(16).
\end{aligned}
\tag{2}
$$

For example, $\mathrm{C}\ell^e(p{+}1,p{+}1) \simeq \mathrm{C}\ell(p,p{+}1) \simeq \mathbb{R}(2^p)\oplus\mathbb{R}(2^p)$.

2.3. $\mathrm{Spin}(p,q)$ **and spinors.** — By the defining relations, if $v\cdot v \neq 0$, then $v \in \mathbb{R}^{p,q}$ is a unit in $\mathrm{C}\ell(p,q)$ and, moreover, the twisted conjugation $\rho(v) : \mathrm{C}\ell(p,q) \to \mathrm{C}\ell(p,q)$

defined on generators $w \in \mathbb{R}^{p,q}$ by $\rho(v)(w) = -vwv^{-1}$ preserves the generating subspace $\mathbb{R}^{p,q} \subset \mathrm{C}\ell(p,q)$, acting as reflection in the hyperplane $v^{\perp} \subset \mathbb{R}^{p,q}$.

The group $\mathrm{Pin}(p,q) \subset \mathrm{C}\ell(p,q)$ is the subgroup of the units in $\mathrm{C}\ell(p,q)$ generated by the elements v where $v \cdot v = \pm 1$ and the group $\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap \mathrm{C}\ell^{e}(p,q)$ is the subgroup of the even Clifford algebra generated by the products vw, where $v{\cdot}v = w{\cdot}w = \pm 1$.

The map ρ defined above extends to a group homomorphism $\rho : \mathrm{Pin}(p,q) \to \mathrm{O}(p,q)$ that turns out to be a non-trivial double cover. The homomorphism $\rho : \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$ is also a non-trivial double cover.

The space of spinors $\mathbb{S}^{p,q}$ is essentially an irreducible $\mathrm{C}\ell(p,q)$-module, considered as a representation of $\mathrm{Spin}(p,q)$.

When $p{-}q \equiv 3 \mod 4$, this definition is independent of which of the two possible irreducible $\mathrm{C}\ell(p,q)$ modules one uses in the construction.

When $p{-}q \equiv 0 \mod 4$, the space $\mathbb{S}^{p,q}$ is a reducible $\mathrm{Spin}(p,q)$-module, in fact, it can be written as a sum $\mathbb{S}^{p,q} = \mathbb{S}^{p,q}_{+} \oplus \mathbb{S}^{p,q}_{-}$ where $\mathbb{S}^{p,q}_{\pm}$ are irreducible. Action by an element of $\mathrm{Pin}(p,q)$ not in $\mathrm{Spin}(p,q)$ exchanges these two summands.

When $p{-}q \equiv 1$ or $2 \mod 8$, the definition of $\mathbb{S}^{p,q}$ as given above turns out to be the sum of two equivalent representations of $\mathrm{Spin}(p,q)$. In this case, it is customary to redefine $\mathbb{S}^{p,q}$ to be one of these two summands, so I do this without comment in the rest of the article.

When $q = 0$, i.e., in the Euclidean case, I will usually simplify the notation by writing $\mathrm{C}\ell(p)$, $\mathrm{Spin}(p)$, and $\mathbb{S}^{p}$ instead of $\mathrm{C}\ell(p,0)$, $\mathrm{Spin}(p,0)$, and $\mathbb{S}^{p,0}$, respectively.

2.4. Orbits in the low dimensions. — I will now describe the $\mathrm{Spin}(p,q)$-orbit structure of $\mathbb{S}^{p,q}$ when $p{+}q \leq 6$. This description made simpler by the fact that there are several 'exceptional isomorphisms' of Lie groups (as discovered by Cartan) that reduce the problem to a series of classical linear algebra problems.

When $p{+}q \leq 1$, these groups are not particularly interesting and, since there is no holonomy in dimension 1 anyway, I will skip these cases.

2.4.1. Dimension 2. — Here there are two cases.

2.4.1.1. $\mathrm{Spin}(2) \simeq \mathrm{U}(1)$. — The action of $\mathrm{Spin}(2) = \mathrm{U}(1)$ on $\mathbb{S}^{2} \simeq \mathbb{C}$ is the unit circle action

$$\lambda \cdot s = \lambda s\,. \tag{3}$$

The orbits of $\mathrm{Spin}(2)$ on $\mathbb{S}^{2} = \mathbb{C}$ are simply the level sets of the squared norm, so all of the nonzero orbits have the same stabilizer, namely, the identity.

Identifying $\mathbb{R}^{2,0}$ with $\mathbb{C}$, the action of $\mathrm{Spin}(2)$ on $\mathbb{R}^{2,0}$ can be described as

$$\lambda \cdot v = \lambda^{2}\, v \tag{4}$$

and the inner product is $v \cdot v = |v|^{2} = \bar{v}\, v$.