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ON THE KATO INEQUALITY IN RIEMANNIAN GEOMETRY

by

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***Abstract.* —** We describe recent works of the authors as well as of T. Branson on refined Kato inequalities for sections of vector bundles living in the kernel of natural first-order elliptic operators

***Résumé* (Sur l'inégalité de Kato en géométrie riemannienne). —** Nous faisons le point sur des travaux récents, dus aux auteurs et aussi à T. Branson, sur des raffinements de l'inégalité de Kato, valables pour des sections d'un fibré vectoriel annulées par un opérateur différentiel naturel et elliptique du premier ordre.

1. Introduction

The Kato inequality is a classical tool in Riemannian geometry. It stands as a useful way to relate vector-valued problems on vector bundles to scalar valued ones involving only functions. It says that for a smooth section ξ of a Riemannian vector bundle E equipped with a compatible connection ∇,

$$\big|d|\xi|\big| \;\leq\; |\nabla\xi|$$

outside the zero-set of ξ. This is an easy consequence of the Schwarz inequality.

More surprisingly, some authors noticed that refined Kato inequalities, of the type

$$\big|d|\xi|\big| \;\leq\; k\,|\nabla\xi| \quad \text{with} \quad k<1\ ,$$

were true for ξ in the kernel of an elliptic first-order differential operator acting on sections of E. This remark was a crucial step in a number of problems involving either decay estimates at infinity of the norm of sections satisfying an elliptic equation (curvature of Einstein metrics on asymptotically flat manifolds, second form of minimal hypersurfaces in spaceforms, Yang-Mills fields on the flat four-space, etc...) or fine-tuned spectral problems.

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The constants k that were found depended strongly on the elliptic operators involved and it was observed that there should exist a systematic way to detect and compute them and that there should be a strong link between their values and representation-theoretic data of the given bundle.

At the time of the meeting in Marseille, we had devised a method leading to computations of optimal refined Kato inequalities in a few cases including all possible situations in dimensions 3 and 4 and a talk on that subject was delivered by the third author. The method was extended shortly after to a systematic one that computes almost all the possible constants and a large number of explicit values were then given [**6**]. During the same period, T. Branson independently found a different method to compute all of them [**5**], based on his earlier works on the spectrum of elliptic second-order differential operators on the round sphere [**4**]. We intend here to report on these two methods, and try to highlight their differences and their relationships. We shall also give a few examples of old and new uses of refined Kato inequalities.

We have tried to make this survey accessible for a reader not acquainted with slightly specialized tools of representation theory (all of which may however be found in the textbook [**8**]). This led us to be somehow imprecise or unspecific at some places in the main body of this text. We thought however that this could be useful for those that were interested rather in the results or the applications of refined Kato inequalities in global analysis on manifolds rather than in the precise course of the proofs. Appendices have been added at the end, containing more elaborate details and precise computations. We then hope that this text may serve as a reading guide before entering the two more technical papers [**5**] and [**6**].

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2. Basics: the classical Kato inequality

We consider from now on an oriented Riemannian manifold M endowed with a vector bundle E induced from a representation of the special orthogonal group $\mathrm{SO}(n)$ or the spin group $\mathrm{Spin}(n)$ (in which case M will be supposed to be spin). If ∇ is any metric connection on E and ξ is any section of E, then

$$2\,\big|d|\xi|\big|\,|\xi| \;=\; \big|d\left(|\xi|^2\right)\big| \;=\; 2\,\big|\langle\nabla\xi,\xi\rangle\big| \;\leq\; 2\,|\nabla\xi|\,|\xi|$$

(with the metric on $T^*M\otimes E$ given by the tensor product metric). Hence we get the classical Kato inequality

$$\big|d|\xi|\big| \;\leq\; |\nabla\xi| \qquad (1)$$

outside the zero set of ξ. Moreover the equality case is achieved at a point if and only if there is a 1-form α such that

$$\nabla\xi \;=\; \alpha \otimes \xi.$$

Following J. P. Bourguignon [**3**], we now consider a section ξ lying in the kernel of a natural first-order operator P on E. Any such operator is the composition of the covariant derivative followed by projection Π on one (or more) irreducible components of the bundle $T^*M \otimes E$, and its symbol reads: $\sigma(P) = \sigma(\Pi \circ \nabla) = \Pi$. Now assume (1) is optimal at some point. The discussion above shows that $\nabla\xi \;=\; \alpha \otimes \xi$ at that point. But

$$0 \;=\; P\xi \;=\; \Pi \circ \nabla\,\xi = \Pi(\alpha \otimes \xi).$$

Thus, optimality is possible if and only if P is not an elliptic operator. Conversely, one might guess that any elliptic operator P gives rise, for any section ξ in its kernel, to a refined Kato inequality

$$|d\xi| \;\leq\; k_P\,|\nabla\xi| \tag{2}$$

with a constant k_P depending only on the operator P involved.

3. Background: conformal weights

We consider an irreducible natural vector bundle E over a Riemannian manifold (M, g) of dimension n, with scalar product $\langle\cdot,\cdot\rangle$ and a metric (not necessarily Levi-Civita) connection ∇. By assumption, E is associated to an irreducible representation λ of the group $\mathrm{SO}(n)$ (resp. $\mathrm{Spin}(n)$ if necessary). The tensor product of λ with the standard representation τ splits in irreducible components as $\tau \otimes \lambda = \oplus_{j=1}^{N}\mu_j$. Equivalently, and to set notations, we write

$$T^*M \otimes E = \oplus_{j=1}^{N} F_j.$$

Projection on the j-th summand will be denoted by Π_j. Apart from the exceptional case where $T^*M \otimes E$ contains two irreducible components for $\mathrm{SO}(n)$ whose sum is an irreducible representation for $\mathrm{O}(n)$, each F_j is an eigenspace for the endomorphism B of $T^*M \otimes E$ defined as

$$B(\alpha \otimes v) = \sum_{i=1}^{n} e_i \otimes (e_i \wedge \alpha)\cdot v$$

where the dot means the action of $\mathfrak{so}(n)$ on the representation space E. The endomorphism B plays an important role in conformal geometry [**9**]. Its eigenvalues are called the *conformal weights*, denoted w_j, and can be easily computed from representation-theoretic data : the Casimir numbers [**8**] of representations λ, τ and

μ_j (normalized as to ensure $C(\mathfrak{so}(n), \tau) = n-1$, see Appendix A for more on this point). More precisely:

$$w_j = \frac{1}{2}\left(C(\mathfrak{so}(n), \mu_j) - C(\mathfrak{so}(n), \lambda) - C(\mathfrak{so}(n), \tau)\right).$$

We shall adopt here the convention *not to split* irreducible representations of $\mathrm{O}(n)$ inside $\tau \otimes \lambda$ into irreducibles for $\mathrm{SO}(n)$. This ensures the conformal weights are *always distinct*, henceforth F_j will always denote the eigenspace associated to w_j, and it corresponds to an irreducible summand of $\tau \otimes \lambda$ except in the exceptional case quoted above where it is a sum of two irreducibles. Moreover, irreducible components will be ordered from 1 to N (the number of distinct eigenspaces) in (strictly) decreasing order of conformal weights (see Appendix A for more details on the representation theory involved).

Since they are easily computable, all the results that follow will be given in terms of the conformal weights, or more precisely in terms of the *modified conformal weights* $\widetilde{w}_j = w_j + (n-1)/2$, eigenvalues of the translated operator $\widetilde{B} = B + (n-1)/2\ \mathrm{id}$.

Natural first order differential are indexed by subsets I of $\{1, \ldots, N\}$. They all are of the following form:

$$P_I = \sum_{i \in I} a_i\, \Pi_i \circ \nabla\ ;$$

any such operator is said to be (injectively, or overdetermined) elliptic if its symbol $\Pi_I = \sum_{i \in I} a_i \Pi_i$ does not vanish on any decomposable element $\alpha \otimes v$ of $T^*M \otimes E$. The coefficients a_i can all be set to 1 without harm as lying in the kernel of the operator is equivalent to lying in the intersection of the kernels of all the elementary operators $P_i = \Pi_i \circ \nabla$ for i in I and being elliptic is equivalent to the fact that no decomposed tensor product lives in the intersection of the kernels of the Π_i.

Elliptic operators in this precise sense have been completely classified by T. Branson in [**4**]. Since any set J containing a subset I such that P_I is elliptic gives rise to an operator P_J which is also elliptic, it suffices to describe the set of *minimal elliptic operators*, *i.e.* the set of operators P_I such that P_J is not elliptic for any proper subset J of I. T. Branson's result provides an explicit description of this set (see Appendix B for more details). For example, the highest weight operator $P_{\{1\}}$ is always minimal elliptic. Moreover and quite surprisingly, sets of indices corresponding to minimal elliptic operators are always small: in fact they contain at most one or two elements.

Our guiding philosophy will now be to prove refined Kato inequalities for sections lying in the kernels of natural first-order elliptic operators on E, with the constants given in terms of the (modified) conformal weights. It is an intersting feature of the problem to note that two genuinely different methods lead to the results. Both end up with semi-explicit expressions of the constants, which can be obtained by solving a minimization problem over a finite set of real numbers. The results can then be made completely explicit in a large number of cases.

The first method, devised by the authors, can be considered as the *local method*. It relies on elaborate algebraic considerations on the conformal weights together with a "linear programming" problem. It is sharp and also provides an explicit description of the sections satisfying equality in the refined Kato inequality at each point. It has the unfortunate feature of being non-sharp for some small (precisely known) set of operators, hopefully seldom encountered in practice.

The second one, or the *global method*, is due to T. Branson. It gives a refined Kato inequality in every case, sharpness is also clear but the equality cases' description is less precise. The proofs rely on the spectral computations on the round sphere done in [**4**] using powerful techniques of harmonic analysis, together with a clever elementary lemma that relates the knowledge of the spectrum of an operator to information on its symbol.

4. Kato constants: linear programming method of computation

The local method finds its roots in the proof of the classical Kato inequality: it aims at obtaining a refined Schwarz inequality for

$$\big|\langle \nabla\xi, \xi\rangle\big|$$

when ξ is a section lying in the kernel of an elliptic first-order operator P_I.

4.1. Ansatz. — Consider Φ an element of $\ker \Pi_I$ at some point (as is $\nabla\xi$ at each point) and v an element of E at the same point (as is ξ). We let I a subset of $\{1,\dots,N\}$, denote by $\widehat{I}$ its complement in $\{1,\dots,N\}$ and compute

$$\begin{aligned}\sup_{|v|=1}\big|\langle\Phi, v\rangle\big| \;&\leq\; \sup_{|\alpha|=|v|=1}\big|\langle\Phi,\alpha\otimes v\rangle\big| \;=\; \sup_{|\alpha|=|v|=1}\big|\langle\Phi,\Pi_{\widehat{I}}(\alpha\otimes v)\rangle\big|\\ &\leq\; \left(\sup_{|\alpha|=|v|=1}|\Pi_{\widehat{I}}(\alpha\otimes v)|\right)|\Phi|.\end{aligned}\tag{3}$$

This gives a refined Kato inequality with $k_I = \sup_{|\alpha|=|v|=1}|\Pi_{\widehat{I}}(\alpha\otimes v)|$. Moreover, equality holds in it if and only if it holds in the refined Schwarz inequality with $v=\xi$ and $\Phi = \nabla\xi$. Hence it is algebraically sharp since the supremum is always attained by compactness. If equality holds, then $\nabla\xi = \Pi_{\widehat{I}}(\alpha\otimes\xi)$ for some $\alpha\otimes\xi$ such that $|\Pi_{\widehat{I}}(\alpha\otimes\xi)|$ is maximal among all $|\Pi_{\widehat{I}}(\alpha\otimes v)|$ with $|\alpha| = |v| = 1$. Moreover such a situation can easily be achieved in the flat case with a suitable affine solution of $P_I\xi = 0$.

4.2. Resolution of the problem. — We now follow the standard method of Lagrange interpolation. Each projection Π_j can be written as

$$\Pi_j \;=\; \prod_{k\neq j}\frac{\widetilde{B}-\widetilde{w}_k\,\mathrm{id}}{\widetilde{w}_j-\widetilde{w}_k} \;=\; \frac{\sum_{k=0}^{N}\widetilde{w}_j^{N-1-k}\left(\sum_{\ell=0}^{k}(-1)^\ell\sigma_\ell(w)\widetilde{B}^{k-\ell}\right)}{\prod_{k\neq j}(\widetilde{w}_j-\widetilde{w}_k)},$$