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EICHLER-SHIMURA RELATIONS AND SEMISIMPLICITY OF ÉTALE COHOMOLOGY OF QUATERNIONIC SHIMURA VARIETIES

BY JAN NEKOVÁŘ

ABSTRACT. – We show that the non CM part of ℓ-adic étale cohomology of any compact quaternionic Shimura variety with coefficients in any automorphic local system is a semisimple Galois representation. If the local system has weight $k = (k_1, \dots, k_d)$ with all k_i of the same parity, the full ℓ-adic étale cohomology is semisimple. For Hilbert modular varieties, analogous results are proved for ℓ-adic intersection cohomology of the Baily-Borel compactification. The proof combines a representation-theoretical criterion of semisimplicity with Eichler-Shimura relations for partial Frobenius morphisms.

RÉSUMÉ. – On montre que l'action galoisienne sur la partie sans multiplication complexe de la cohomologie étale d'un faisceau ℓ-adique lisse automorphe sur une variété de Shimura quaternionique compacte est semi-simple. Si le poids du faisceau s'écrit $k = (k_1, \dots, k_d)$, où les k_i ont la même parité, toute la cohomologie étale est semi-simple. Les mêmes résultats sont montrés pour la cohomologie d'intersection ℓ-adique de la compactification de Baily-Borel des variétés modulaires de Hilbert. La preuve utilise un critère abstrait de semi-simplicité et les relations d'Eichler-Shimura pour les morphismes de Frobenius partiels.

0. Introduction

0.1. General conventions and notation

The characteristic polynomial of an endomorphism u of a finite-dimensional vector space over a field k will be denoted by $P_u(X) = \det(X \cdot \mathrm{id} - u) \in k[X]$. If $k \subset K$ are fields and X is a k-vector subspace of a K-vector space Y, we denote by $K \cdot X$ the K-vector subspace of Y generated by X. We abbreviate $\otimes_{\mathbf{Z}}$ as $\otimes$. For an abelian group A we let $\widehat{A} = A \otimes \widehat{\mathbf{Z}}$. We denote by $\mathbf{A}$ and $\mathbf{A}_k$, respectively, the ring of adeles of $\mathbf{Q}$ and of a number field k.

Throughout the article we fix an isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell$. For any algebraic object $(-)$ defined over a subfield of $\mathbf{C}$ we denote by $(-)_\ell$ its base change to $\overline{\mathbf{Q}}_\ell$. Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. The reciprocity map of class field theory is normalized by letting uniformisers correspond to geometric Frobenius elements $\mathrm{Fr}(P)$. All representations and

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characters are assumed to be continuous with respect to the natural topologies involved. A representation of a profinite group is called strongly irreducible if its restriction to every open subgroup is irreducible. By an automorphic representation we mean an irreducible automorphic representation.

0.2. – Let us recall basic facts about decomposition of singular and étale cohomology of compact Shimura varieties. As in the classical case of cuspidal cohomology of modular curves, everything boils down to the fact that the Hecke operators act on the space of cuspidal automorphic forms in a selfadjoint way (up to a twist).

Let $(G, \mathscr{X})$ be a (pure) Shimura datum. A rational representation $\xi : G_{\mathbf{C}} \longrightarrow GL(N)_{\mathbf{C}}$ (whose restriction to the center $Z_{\mathbf{C}}$ satisfies an appropriate condition) gives rise, for each sufficiently small open compact subgroup $K \subset G(\widehat{\mathbf{Q}})$, to a locally constant sheaf of complex vector spaces $\mathscr{L}_\xi$ on the complex manifold $\mathrm{Sh}_K(G, \mathscr{X})^{\mathrm{an}} = G(\mathbf{Q})\backslash(\mathscr{X} \times G(\widehat{\mathbf{Q}})/K)$.

0.3. – If, in addition, the derived group G^{der} is anisotropic, then $\mathrm{Sh}_K(G, \mathscr{X})^{\mathrm{an}}$ is compact and its cohomology $H^*(\mathrm{Sh}_K(G, \mathscr{X})^{\mathrm{an}}, \mathscr{L}_\xi)$ is described in terms of relative Lie algebra cohomology.

Write $\mathscr{X} = G(\mathbf{R})/K_\infty$, where K_∞ is the stabilizer of a fixed base point in $\mathscr{X}$, and denote, for any $G(\mathbf{R})$-module V, by V_0 the subspace of K_∞-finite vectors in V. There is a canonical isomorphism

$$\begin{aligned} &H^i(\mathrm{Sh}_K(G, \mathscr{X})^{\mathrm{an}}, \mathscr{L}_\xi) \\ &\quad = H^i(\mathfrak{g}, K_\infty; C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K) \otimes \xi) = H^i(\mathfrak{g}, K_\infty; C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K)_0 \otimes \xi), \end{aligned}$$

where $\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R}))$ [4, VII.2.7]. It gives rise to a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

(0.3.1)
$$H^i(\mathrm{Sh}(G, \mathscr{X})^{\mathrm{an}}, \mathscr{L}_\xi) = \varinjlim_K H^i(\mathrm{Sh}_K(G, \mathscr{X})^{\mathrm{an}}, \mathscr{L}_\xi) = H^i(\mathfrak{g}, K_\infty; C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A}))_0 \otimes \xi).$$

For every character $\omega : Z(\mathbf{Q})\backslash Z(\mathbf{A}) \longrightarrow \mathbf{C}^\times$ fix a character $\omega' : G(\mathbf{Q})\backslash G(\mathbf{A}) \longrightarrow \mathbf{R}_+^\times$ such that $\omega'|_{Z(\mathbf{A})} = |\omega|$. The space $G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})$ is compact (since G^{der} is anisotropic) and the completion $L^2(G, \omega)$ of

$$C^\infty(G, \omega) = \{f \in C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})) \mid f(gz) = \omega(z) f(g) \ \ \forall z \in Z(\mathbf{A})\}$$

with respect to the norm

$$|f|^2 = \int_{G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})} (\omega'(g)^{-1}|f(g)|)^2 \, dg$$

is a unitary representation of $G(\mathbf{A})$ under the action $(g * f)(h) = \omega'(g)^{-1} f(hg)$. This representation decomposes as a discrete Hilbert sum $L^2(G, \omega) = \widehat{\bigoplus} m(\pi')\,\pi'$ of unitary automorphic representations π' of $G(\mathbf{A})$ with finite multiplicities $m(\pi')$.

Each π' has central character $\omega_{\pi'} = \omega/|\omega|$ and gives rise to an automorphic representation $\pi = \omega'\pi' = \pi_\infty \otimes \pi^\infty$ of $G(\mathbf{A}) = G(\mathbf{R}) \times G(\widehat{\mathbf{Q}})$ with central character $\omega_\pi = \omega$.

Matsushima's formula [4, Thm. VII.5.2] yields a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

$$H^i(\mathfrak{g}, K_\infty; C^\infty(G, \omega)_0 \otimes \xi) = \bigoplus_{\pi = \pi_\infty \otimes \pi^\infty} m(\pi)\, H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \otimes \pi^\infty,$$