ASTÉRISQUE 305

H^∞ FUNCTIONAL CALCULUS AND SQUARE FUNCTIONS ON NONCOMMUTATIVE L^p-SPACES

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H^∞ FUNCTIONAL CALCULUS AND SQUARE FUNCTIONS ON NONCOMMUTATIVE L^p-SPACES

Marius Junge, Christian Le Merdy, Quanhua Xu

***Abstract.* —** We investigate sectorial operators and semigroups acting on noncommutative L^p-spaces. We introduce new square functions in this context and study their connection with H^∞ functional calculus, extending some famous work by Cowling, Doust, McIntoch and Yagi concerning *commutative* L^p-spaces. This requires natural variants of Rademacher sectoriality and the use of the matricial structure of noncommutative L^p-spaces. We mainly focus on noncommutative diffusion semigroups, that is, semigroups $(T_t)_{t\geq 0}$ of normal selfadjoint operators on a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ such that $T_t\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a contraction for any $p \geq 1$ and any $t \geq 0$. We discuss several examples of such semigroups for which we establish bounded H^∞ functional calculus and square function estimates. This includes semigroups generated by certain Hamiltonians or Schur multipliers, q-Ornstein-Uhlenbeck semigroups acting on the q-deformed von Neumann algebras of Bozejko-Speicher, and the noncommutative Poisson semigroup acting on the group von Neumann algebra of a free group.

Résumé **(Calcul fonctionnel H^∞ et fonctions carrées dans les espaces L^p non commutatifs)**

Nous étudions les opérateurs sectoriels et les semigroupes opérant sur un espace L^p non commutatif. Nous introduisons de nouvelles fonctions carrées adaptées à ce contexte et étudions leurs interactions avec le calcul fonctionnel H^∞. Nous obtenons des extensions de travaux fameux de Cowling, Doust, McIntoch et Yagi qui concernaient le cas commutatif. Cette étude nécessite l'introduction de variantes de la Rademacher sectorialité et l'usage des structures matricielles sur les espaces L^p non commutatifs. Nous traitons de façon approfondie les semigroupes de diffusion non commutatifs. Il s'agit des semigroupes $(T_t)_{t\geq 0}$ d'opérateurs normaux et auto-adjoints opérant sur une algèbre de von Neumann semifinie $(\mathcal{M},\tau)$, tels que $T_t\colon L^p(\mathcal{M})\to L^p(\mathcal{M})$ est une contraction pour tout $p\geq 1$ et pour tout $t\geq 0$. Nous présentons et étudions plusieurs exemples de tels semigroupes, pour lesquels nous sommes en mesure d'établir une propriété de calcul H^∞ borné, ainsi que des estimations quadratiques. Cette étude inclut certains semigroupes engendrés par des opérateurs Hamiltoniens ou des multiplicateurs de Schur, des semigroupes d'Ornstein-Uhlenbeck opérant sur les algèbres de von Neumann de q-déformation de Bozejko-Speicher, et le semigroupe de Poisson non commutatif défini sur l'algèbre de von Neumann d'un groupe libre.

CONTENTS

CHAPTER 1

INTRODUCTION

In the recent past, noncommutative analysis (in a wide sense) has developed rapidly because of its interesting and fruitful interactions with classical theories such as C^*-algebras, Banach spaces, probability, or harmonic analysis. The theory of operator spaces has played a prominent role in these developments, leading to new fields of research in either operator theory, operator algebras or quantum probability. The recent theory of martingale inequalities in noncommutative L^p-spaces is a good example for this development. Indeed, square functions associated to martingales and most of the classical martingale inequalities have been successfully transferred to the noncommutative setting. See in particular [**64, 33, 68, 38**], and also the recent survey [**80**] and the references therein. The noncommutative maximal ergodic theorem in [**36**] is our starting point for the study of noncommutative diffusion semigroups. On this line we investigate noncommutative analogs of classical square function inequalities.

It is remarkable that operator space techniques have led to new results on classical analysis. We mention in particular completely bounded Fourier multipliers and Schur multipliers on Schatten classes [**31**]. In our treatment of semigroups no prior knowledge on operator space theory is required. However, operator space concepts underlie our understanding of the subject.

Our objectives are to introduce natural square functions associated with a sectorial operator or a semigroup on some noncommutative L^p-space, to investigate their connections with H^∞ functional calculus, and to give various concrete examples and applications. H^∞ functional calculus was introduced by McIntosh [**53**], and then developed by him and his coauthors in a series of remarkable papers [**54, 21, 3**]. Nowadays this is a classical and powerful subject which plays an important role in spectral theory for unbounded operators, abstract maximal L^p-regularity, or multiplier theory. See e.g. [**43**] for more information and references.

Square functions for generators of semigroups appeared earlier in Stein's classical book [**70**] on the Littlewood-Paley theory for semigroups acting on usual (=commutative) L^p-spaces. Stein gave several remarkable applications of these square functions to functional calculus and multiplier theorems for diffusion semigroups. Later on, Cowling [**20**] obtained several extensions of these results and used them to prove maximal theorems.

The fundamental paper [**21**] established tight connections between McIntosh's H^∞ functional calculus and Stein's approach. Assume that A is a sectorial operator on $L^p(\Sigma)$, with $1 < p < \infty$, and let F be a non zero bounded analytic function on a sector $\{|\mathrm{Arg}(z)| < \theta\}$ containing the spectrum of A, and such that F tends to 0 with an appropriate estimate as $|z| \to \infty$ and as $|z| \to 0$ (see Chapter 3 for details). The associated square function is defined by

$$\|x\|_F = \Big\|\Big(\int_0^\infty \big|F(tA)x\big|^2 \,\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_p, \qquad x \in L^p(\Sigma).$$

For example if $-A$ is the generator of a bounded analytic semigroup $(T_t)_{t\geq 0}$ on $L^p(\Sigma)$, then we can apply the above with the function $F(z) = ze^{-z}$ and in this case, we obtain the familiar square function

$$\|x\|_F = \Big\|\Big(\int_0^\infty t\,\Big|\frac{\partial}{\partial t}\big(T_t(x)\big)\Big|^2\, dt\Big)^{\frac{1}{2}}\Big\|_p$$

from [**70**, Chapters III-IV]. One of the most remarkable connections between H^∞ functional calculus and square functions on L^p-spaces is as follows. If A admits a bounded H^∞ functional calculus, then we have an equivalence $K_1\|x\| \leq \|x\|_F \leq K_2\|x\|$ for any F as above. Indeed this follows from [**21**] (see also [**49**]).

In this paper we consider a sectorial operator A acting on a noncommutative L^p-space $L^p(\mathcal{M})$ associated with a semifinite von Neumann algebra $(\mathcal{M},\tau)$. For an appropriate bounded analytic function F as before, we introduce two square functions which are *approximately* defined as

$$\|x\|_{F,c} = \Big\|\Big(\int_0^\infty \big(F(tA)x\big)^*\big(F(tA)x\big)\,\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_p$$

and

$$\|x\|_{F,r} = \Big\|\Big(\int_0^\infty \big(F(tA)x\big)\big(F(tA)x\big)^*\,\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_p$$

(see Chapter 6 for details). The functions $\|\ \|_{F,c}$ and $\|\ \|_{F,r}$ are called column and row square functions respectively. Using them we define a symmetric square function $\|x\|_F$. As with the noncommutative Khintchine inequalities (see [**51, 52**]), this definition depends upon whether $p \geq 2$ or $p < 2$. If $p \geq 2$, we set $\|x\|_F = \max\{\|x\|_{F,c}\,; \|x\|_{F,r}\}$. See Chapter 6 for the more complicated case $p < 2$. Then one of our main results is that if A admits a bounded H^∞ functional calculus on $L^p(\mathcal{M})$, with $1 < p < \infty$, we have an equivalence

$$K_1\|x\| \leq \|x\|_F \leq K_2\|x\| \tag{1.1}$$

for these square functions.

After a short introduction to noncommutative L^p-spaces, Chapter 2 is devoted to preliminary results on noncommutative Hilbert space valued L^p-spaces, which are central for the definition of square functions. These spaces and related ideas first appeared in [**51**] (see also [**52, 62**]). In fact operator valued matrices and operator space techniques (see e.g. [**58, 62, 63**]) play a natural role in our context. However we tried to make the paper accessible to readers not familiar with operator space theory and completely bounded maps.

In Chapter 3 we give the necessary background on sectorial operators, semigroups, and H^∞ functional calculus. Then we introduce a completely bounded H^∞ functional calculus for an operator A acting on a noncommutative $L^p(\mathcal{M})$. Again this is quite natural in our context and indeed it turns out to be important in our study of square functions (see in particular Corollary 7.9).

Rademacher boundedness and Rademacher sectoriality now play a prominent role in H^∞ functional calculus. We refer the reader e.g. to [**41**], [**79**], [**78**], [**47**], [**49**] or [**43**] for developments and applications. On noncommutative L^p-spaces, it is natural to introduce two related concepts, namely the column boundedness and the row boundedness. If $\mathcal{F}$ is a set of bounded operators on $L^p(\mathcal{M})$, we will say that $\mathcal{F}$ is Col-bounded if we have an estimate

$$\Big\|\Big(\sum_k T_k(x_k)^*T_k(x_k)\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})} \leq C\,\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})}$$

for any finite families $T_1,\ldots,T_n$ in $\mathcal{F}$, and $x_1,\ldots,x_n$ in $L^p(\mathcal{M})$. Row boundedness is defined similarly. We develop these concepts in Chapter 4, along with the related notions of column and row sectoriality.

Chapters 6 and 7 are devoted to square functions and their interplay with H^∞ functional calculus. As a consequence of the main result of Chapter 4, we prove that if A is Col-sectorial (resp. Rad-sectorial), then we have an equivalence

$$K_1\|x\|_{G,c} \leq \|x\|_{F,c} \leq K_2\|x\|_{G,c} \qquad (\text{resp. } K_1\|x\|_G \leq \|x\|_F \leq K_2\|x\|_G\,)$$

for any pair of non zero functions F, G defining square functions. This is a noncommutative generalization of the main result of [**49**]. Then we prove the aforementioned result that (1.1) holds true if A has a bounded H^∞ functional calculus. We also show that conversely, appropriate square function estimates for an operator A on $L^p(\mathcal{M})$ imply that A has a bounded H^∞ functional calculus.

Chapter 5 (which is independent of Chapters 6 and 7) is devoted to a noncommutative generalization of Stein's diffusion semigroups considered in [**70**]. We define a noncommutative diffusion semigroup to be a point w^*-continuous semigroup $(T_t)_{t\geq 0}$ of normal contractions on $(\mathcal{M},\tau)$, such that each T_t is selfadjoint with respect to τ. In this case, $(T_t)_{t\geq 0}$ extends to a c_0-semigroup of contractions on $L^p(\mathcal{M})$ for any $1\leq p<\infty$. Let $-A_p$ denote the negative generator of the L^p-realization of $(T_t)_{t\geq 0}$. Our main result in this chapter is that if further each $T_t\colon \mathcal{M}\to\mathcal{M}$ is positive (resp. completely positive), then A_p is Rad-sectorial (resp. Col-sectorial and Row-sectorial). The proof is based on a noncommutative maximal theorem from [**37, 36**], where such diffusion semigroups were considered for the first time.

If $(T_t)_{t\geq 0}$ is a noncommutative diffusion semigroup as above, the most interesting general question is whether A_p admits a bounded H^∞ functional calculus on $L^p(\mathcal{M})$ for $1<p<\infty$. This question has an affirmative answer in the commutative case [**20**] but it is open in the noncommutative setting. The last three chapters are devoted to examples of natural diffusion semigroups, for which we are able to show that A_p admits a bounded H^∞ functional calculus. Here is a brief description.

In Chapter 8, we consider left and right multiplication operators, Hamiltonians, and Schur multipliers on Schatten space S^p. Let H be a real Hilbert space, and let $(\alpha_k)_{k\geq 1}$ and $(\beta_k)_{k\geq 1}$ be two sequences of H. We consider the semigroup $(T_t)_{t\geq 0}$ of Schur multipliers which are determined by $T_t(E_{ij})=e^{-t(\|\alpha_i-\beta_j\|)}E_{ij}$, where the E_{ij}'s are the standard matrix units. This is a diffusion semigroup on $B(\ell^2)$ and we show that the associated negative generators A_p have a bounded H^∞ functional calculus for any $1<p<\infty$.

Let H be a real Hilbert space. In Chapter 9, we consider the q-deformed von Neumann algebras $\Gamma_q(H)$ of Bozejko and Speicher [**14, 15**], equipped with its canonical trace. To any contraction $a\colon H\to H$ we may associate a second quantization operator $\Gamma_q(a)\colon \Gamma_q(H)\to\Gamma_q(H)$, which is a normal unital completely positive map. We consider semigroups defined by $T_t=\Gamma_q(a_t)$, where $(a_t)_{t\geq 0}$ is a selfadjoint contraction semigroup on H. This includes the q-Ornstein-Uhlenbeck semigroup [**9, 11**]. These semigroups $(T_t)_{t\geq 0}$ are completely positive noncommutative diffusion semigroups and we show that the associated A_p's have a bounded H^∞ functional calculus for any $1<p<\infty$.

In Chapter 10 we consider the noncommutative Poisson semigroup of a free group. Let $G=\mathbb{F}_n$ be the free group with n generators and let $|\cdot|$ be the usual length function on G. Let $VN(G)$ be the group von Neumann algebra and let $\lambda(g)\in VN(G)$ be the left translation operator for any $g\in G$. For any $t\geq 0$, T_t is defined by $T_t(\lambda(g))=e^{-t|g|}\lambda(g)$. This semigroup was introduced by Haagerup [**30**]. Again this is a completely positive noncommutative diffusion semigroup and we prove that the associated A_p's have a bounded H^∞ functional calculus for any $1<p<\infty$. The proof uses noncommutative martingales in the sense of [**64**], and we establish new square function estimates of independent interest for these martingales.