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THE DYNAMICS OF GENERIC KUPERBERG FLOWS

Steven HURDER and Ana RECHTMAN

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THE DYNAMICS OF GENERIC KUPERBERG FLOWS

by Steven HURDER and Ana RECHTMAN

Abstract. — In this work, we study the dynamical properties of Krystyna Kuperberg's aperiodic flows on 3-manifolds. We introduce the notion of a "zippered lamination," and with suitable generic hypotheses, show that the unique minimal set for such a flow is an invariant zippered lamination. We obtain a precise description of the topological and dynamical properties of the minimal set, including the presence of non-zero entropy-type invariants and chaotic behavior. Moreover, we show that the minimal set does not have stable shape, yet satisfies the Mittag-Leffler condition for homology groups.

***Résumé* (Dynamique des flots de Kuperberg génériques.)** — Dans ce travail, nous étudions les propriétés dynamiques des flots sans orbites périodiques construits par Krystyna Kuperberg sur les variétés de dimension 3. Nous introduisons la notion de « lamination à fermeture éclair » et, sous des hypothèses de généricité, nous montrons que l'unique ensemble minimal de ces flots est une lamination à fermeture éclair invariante. Nous donnons une description précise de la topologie et des propriétés dynamiques de l'ensemble minimal, parmi lesquelles la présence de phénomènes d'entropie nulle ainsi que du comportement chaotique. Finalement, nous prouvons que l'ensemble minimal a une forme instable au sens de la théorie de la forme, et satisfait la condition de Mittag-Leffler pour les groupes d'homologie d'une suite de voisinages.

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CHAPTER 1

INTRODUCTION

The "Seifert Conjecture," as originally formulated in 1950 by Seifert in [**42**], asked: "Does every non-singular vector field on the 3-sphere $\mathbb{S}^3$ have a periodic orbit?" The partial answers to this question have a long history. F.W. Wilson constructed in the 1966 work [**47**] a smooth flow on a *plug* with exactly two periodic orbits, which was used to modify a given flow on a 3-manifold to obtain one with only isolated periodic orbits. Paul Schweitzer showed in the 1974 work [**41**], that for any closed 3-manifold M, there exists a non-singular C^1-vector field on M without periodic orbits. Schweitzer's result suggested a modified version of Seifert's question: "Does every non-vanishing C^∞ vector field on a closed 3-manifold have a periodic orbit?" Krystyna Kuperberg showed in her celebrated 1994 work [**26**], that the smooth Seifert Conjecture is also false by inventing a construction of aperiodic plugs which is renowned for its simplicity, beauty and subtlety.

***Theorem 1.1* (K. Kuperberg).** — *On every closed oriented* 3*-manifold* M*, there exists a* C^∞ *non-vanishing vector field without periodic orbits.*

The goal of this work is to understand the dynamical properties of such "Kuperberg flows," especially the structure of their minimal sets. In the exploration of these properties, we reveal their beauty and discover the hidden complexity of the Kuperberg dynamical systems.

Let us recall the strategy of the proofs of the results cited above. A *plug* is a compact 3-manifold with boundary in $\mathbb{R}^3$, equipped with a flow satisfying additional conditions. The flow in a plug is assumed to be parallel to the "vertical" part of the boundary, so that it may be inserted in any coordinate chart of a 3-manifold M to modify the given flow on M, and changes only those orbits entering and leaving the "horizontal" faces of the plug. Another assumption on the flow in a plug is that there are orbits, which are said to be *trapped*, which enter the plug and never exit. The closure of such an orbit limits to a compact invariant set contained entirely within the interior of the plug, thus the plug must contain at least one minimal set. In the case of the Wilson Plug, the two periodic orbits are the minimal sets.

A plug is said to be *aperiodic* if it contains no closed orbits. Schweitzer observed in [**41**] that the role of the periodic orbits in a Wilson Plug could be replaced by Denjoy minimal sets, resulting in an aperiodic plug, which could then be used to "open up" the isolated closed orbits provided by Wilson's result. The flow in the Schweitzer Plug is only C^1, due to the topology of the minimal set contained in the plug, around which all trapped orbits for the flow must accumulate. Harrison constructed in [**19**] a modified "non-flat" embedding of the Denjoy continuum into a 3-ball, which she used to construct an aperiodic plug with a C^2-flow. In contrast, Handel showed in [**18**] that if the trapped orbits of a plug accumulate on a minimal set whose topological dimension is one and is the only *invariant* set for the flow in the plug, then the minimal set is *surface-like*: the flow restricted to the minimal set is topologically conjugated to the minimal set of a flow on a surface.

Kuperberg's construction in [**26**] of aperiodic smooth flows on plugs introduced a fundamental new idea, that of "geometric surgery" on a modified version of the Wilson Plug $\mathbb{W}$, to obtain the *Kuperberg Plug* $\mathbb{K}$ as a quotient space, $\tau\colon \mathbb{W} \to \mathbb{K}$. The Wilson vector field $\mathcal{W}$ on $\mathbb{W}$ is modified to provide a smooth vector field $\mathcal{K}$ on the quotient. The flow of $\mathcal{K}$ is denoted by Φ_t. This is said to be a *Kuperberg flow* on $\mathbb{K}$.

The periodic orbits for the Wilson flow on $\mathbb{W}$ get "cut-open" when they are mapped to $\mathbb{K}$, and there they become trapped orbits for Φ_t. The essence of the novel strategy behind the aperiodic property of Φ_t is perhaps best described by a quote from the paper by Matsumoto [**32**]:

> We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to *let closed orbits demolish themselves.* We set up a trap within enemy lines and watch them settle their dispute while we take no active part.

The images in $\mathbb{K}$ of the cut-open periodic orbits from the Wilson flow Ψ_t on $\mathbb{W}$, generate two orbits for the Kuperberg flow Φ_t on $\mathbb{K}$, which are called the *special orbits* for Φ_t. These two special orbits play an absolutely central role in the study of the dynamics of a Kuperberg flow.

There followed after Kuperberg's seminal work, a collection of three works explaining in further detail the proof of the aperiodicity for the Kuperberg flow, and investigating its dynamical properties:

— the Séminaire Bourbaki lecture [**17**] by Étienne Ghys;
— the notes by Shigenori Matsumoto [**32**] in Japanese, translated into English;
— the joint paper [**25**] by Greg Kuperberg and Krystyna Kuperberg.

It was observed in these works that the special orbits in $\mathbb{K}$ each limit to the other, and that a Kuperberg flow has a unique minimal set, which we denote by Σ. The topological and dynamical properties of the minimal sets for the Wilson, Schweitzer and Harrison Plugs are fundamental aspects of the constructions of the flows in these plugs. For the Kuperberg flow, the minimal set Σ is not specified by the construction, but rather its topological properties are a consequence of the strong interaction of