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Abstract. — In [**96**], Voevodsky defines a triangulated category $\mathsf{DM}(k)$ and shows that this category possesses many of the properties that the derived category of a conjectural category of mixed motives would satisfy. One unsatisfactory aspect of his theory was that most of the results about non-smooth varieties, for example that higher Chow groups appear as hom groups in $\mathsf{DM}(k)$, were only available over a characteristic zero base field.

Working with $\mathbb{Z}[1/p]$-coefficients, we obtain all the results of Friedlander, Suslin, and Voevodsky in [**96**] over a perfect field of exponential characteristic p.

The strategy is to replace Voevodsky's application of resolution of singularities via the cdh-topology, with Gabber's theorem on alterations, *cf.* [**40**], via a refinement of the cdh-topology which we christen the $l$dh-topology. As Gabber's theorem only provides an analogue of the existence of a desingularisation, and not the weak factorisation theorem, the proof of cdh-descent of [**19**] cannot be adapted. We instead use a completely different strategy, proving cdh-descent using Ayoub's proper base change theorem, [**3**], for the stable homotopy category of Morel and Voevodsky.

Résumé **(Motifs de Voevodsky et descente pour la topologie *l*dh)**

Dans [**96**], Voevodsky définit une catégorie triangulée $\mathsf{DM}(k)$ et démontre que cette catégorie est douée de plusieurs propriétés que la catégorie dérivée de la catégorie conjecturale des motifs mixtes devrait satisfaire. Un aspect peu satisfaisant de sa théorie est que la plupart des résultats sur les variétés singulières, par exemple le calcul des groupes de Chow supérieurs comme groupes de morphismes dans $\mathsf{DM}(k)$, n'est valide que sur un corps de base de caractéristique zéro. En travaillant avec des coefficients $\mathbb{Z}[1/p]$-linéaires, nous obtenons tous les résultats de Friedlander, Suslin, et Voevodsky de [**96**] sur un corps parfait de caractéristique exponentielle p.

La stratégie consiste à remplacer l'application par Voevodsky de la résolution des singularités via la topologie cdh, par un théorème de Gabber sur les altérations, cf. [**40**], via un raffinement de la topologie cdh que nous baptisons topologie $l$dh. Comme le théorème de Gabber ne fournit qu'un analogue de l'existence d'une désingularisation, mais pas le théorème de factorisation faible, la preuve de la descente cdh de [**19**] ne s'adapte pas à cette situation. Nous montrons la descente cdh avec une stratégie complètement différente qui passe par le théorème de changement de base propre d'Ayoub [**3**] pour la catégorie homotopique stable à la Morel et Voevodsky.

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CHAPTER 1

INTRODUCTION

Main result and some applications. — In [**88**], Voevodsky introduced a triangulated category of motives, and using difficult theorems from [**19**], [**87**], [**79**], and [**78**], proved a number of important results about this category. An annoying obstacle in positive characteristic however, was that to prove anything about motives of non-smooth varieties, a strong version of resolution of singularities needed to be assumed, see (Ros1) and (Ros2) below. The proof of duality for smooth varieties, [**88**, Thm. 4.3.7 (3)], also required this assumption, as did the following theorem of Suslin.

Theorem 1.0.1 (cf. [**78**, Intro.]) — *Let k be an algebraically close field, m an integer invertible in k, let X be a separated finite type k-scheme, and let $j \leq 0$. Then if k satisfies* (Ros1) *and* (Ros2) *below, there exists a natural isomorphism*

$$H_c^{n+2j}(X, \mathbb{Z}/m(j)) \cong \mathrm{CH}_j(X, n; \mathbb{Z}/m)^{\#}$$

where H_c is étale cohomology with compact supports, CH *are Bloch's higher Chow groups, and $(-)^{\#} = \hom_{\mathrm{Ab}}(-, \mathbb{Q}/\mathbb{Z})$.*

In this article we prove:

Theorem A (Chapters 4 and 5). — *All the results in* [**88**] *and* [**19**] *hold true over a perfect field of exponential characteristic p, without assuming that resolution of singularities holds, if $\mathbb{Z}[\frac{1}{p}]$ coefficients are used.*

In particular, this implies that in the theorem of Suslin above, the hypotheses (Ros1) and (Ros2) can be removed. For details about this latter see [**48**]. Theorem A has also been used by Friedlander and Ross in [**18**] to study cycle based perversity homology and cohomology theories, by Hoyois, the author, and Østvær in [**37**] to study the motivic Steenrod algebra, by Kahn and Yamazaki in [**45**] to compare a generalisation of Kato and Somekawa's K-groups with higher Chow groups, by Bondarko and Sosnilo in [**7**] to vastly generalise decomposition of the diagonal type statements, by Cisinski and Déglise in [**10**] to equip $\mathbb{Z}[\frac{1}{p}]$-linear categories of mixed motives with a six functor formalism and show they agree with Voevodsky's categories over regular bases, by Kohrita in [**53**] to study the algebraic part of motivic cohomology–the part parametrisable by semi-abelian varieties, by Kohrita in [**54**] to study some negative motivic homology groups, by Barbieri-Viale and Kahn in [**4**] to study the motivic Albanese and Picard complexes, by the author and Saito in [**50**] to study weight homology theories, by Kerz, Esnault, and Wittenberg in [**51**] to study 1-cycles on projective schemes over discrete valuation rings, and by Totaro in [**84**] to study Künneth properties of Chow rings of classifying spaces of affine group schemes of finite type over a field.

The problem. — A glance at [**88**, §4] shows that the fulcrum of the theory for non-smooth varieties is [**88**, Thm. 4.1.2]. Once one has this theorem everything else falls into place almost by itself. Our version of [**88**, Thm. 4.1.2] is the following.

Theorem B (Theorem 4.0.1, *cf.* [**88**, Thm. 4.1.2], [**19**, Thm. 5.5 (2)])

Let k be a perfect field of exponential characteristic p. Let F be a presheaf of $\mathbb{Z}[\frac{1}{p}]$-modules with transfers on $\mathsf{Sm}(k)$ such that for any smooth scheme X over k and a section $\phi \in F(X)$ there is a proper birational morphism $p : X' \to X$ with $p^*(\phi) = 0$. Then the Suslin complex[1] *$\underline{C}_*(F)$ is quasi-isomorphic to zero, as a complex of Zariski sheaves on $\mathsf{Sm}(k)$.*

The Friedlander-Voevodsky proof of the integral coefficient version of Theorem B assumes resolution of singularities in the following sense.

Definition 1.0.2 (*cf.* [**19**, Def. 3.4]). — Let k be a field. We say that k admits resolution of singularities if the following two conditions hold:

(RoS1) For any scheme of finite type X over k there is a proper surjective morphism $Y \to X$ such that Y is smooth over k, and which is an isomorphism over a dense open subscheme of X with its reduced scheme structure.

(RoS2) For any smooth schemes X, X' over k, and a proper birational morphism $q : X' \to X$ there exists a sequence of blow-ups $p : X_n \to \cdots \to X_1 = X$ with smooth centres such that p factors through q.

We would like to replace this assumption with a recent theorem of Gabber on alterations. Here is a weak version, which is sufficient for working over perfect fields.

Theorem G (*cf.* [**40**, Thm. 3 (1)]). — *Let k be a perfect field of exponential characteristic p, and $l \neq p$ a prime. For any reduced scheme of finite type X over k there is a proper surjective morphism $f : Y \to X$ such that Y is smooth over k, and such that over a dense open subscheme of X, the morphism f is a finite flat of degree prime to l.*

1. $\underline{C}_*F$ is the complex $(\underline{C}_nF)(X) = F(\Delta^n \times X)$ where $\Delta^n \subset \mathbb{A}^{n+1} = \operatorname{Spec}(k[t_0, \ldots, t_n])$ is defined by the equation $\sum t_i = 1$ and differentials $F(\Delta^n \times X) \to F(\Delta^{n-1} \times X)$ are the alternating sums of the morphisms induced by the "face" morphisms $\Delta^{n-1} \to \Delta^n; t_i \mapsto 0$.