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**PERIODS AND HARMONIC ANALYSIS
ON SPHERICAL VARIETIES**

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PERIODS AND HARMONIC ANALYSIS ON SPHERICAL VARIETIES

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Abstract. — Given a spherical variety X for a group G over a non-Archimedean local field k , the Plancherel decomposition for $L^2(X)$ should be related to “distinguished” Arthur parameters into a dual group closely related to that defined by Gaitsgory and Nadler. Motivated by this, we develop, under some assumptions on the spherical variety, a Plancherel formula for $L^2(X)$ up to discrete (modulo center) spectra of its “boundary degenerations”, certain G -varieties with more symmetries which model X at infinity. Along the way, we discuss the asymptotic theory of subrepresentations of $C^\infty(X)$ and establish conjectures of Ichino-Ikeda and Lapid-Mao. We finally discuss global analogues of our local conjectures, concerning the period integrals of automorphic forms over spherical subgroups.

Résumé. — Ce volume développe l'idée selon laquelle l'analyse harmonique d'une variété sphérique X est étroitement liée au programme de Langlands. Dans le cas local, la conjecture principale dit que la décomposition spectrale de $L^2(X)$ est contrôlée par un groupe dual attaché à X . En poursuivant cette idée, les auteurs établissent une formule de Plancherel pour $L^2(X)$, faisant intervenir des variétés sphériques plus simples qui apparaissent dans la géométrie du bord de X . Cette étude locale est ensuite reliée aux conjectures globales sur les périodes de formes automorphes le long de sous-groupes sphériques.

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