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L-GROUPS AND THE LANGLANDS PROGRAM
FOR COVERING GROUPS

Wee Teck Gan, Fan Gao & Martin H. Weissman

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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS

Wee Teck Gan, Fan Gao & Martin H. Weissman

Abstract. — This volume proposes an extension of the Langlands program to covers of quasisplit groups, where covers are those that arise from central extensions of reductive groups by K_2 . By constructing an L-group for any such cover, one may conjecture a parameterization of genuine irreducible representations by Langlands parameters. Two constructions of the L-group are given, and related to each other in a final note. The proposed local Langlands conjecture for covers (LLCC) is proven for covers of split tori, spherical representations in the p-adic case, and discrete series for double-covers of real semisimple groups. The introduction of the L-group allows one to define partial L-functions and functoriality, including base change, for representations of covering groups.

Résumé (L-groupes et le programme de Langlands pour les revêtements de groupes réductifs.) — Ce volume propose une extension du programme de Langlands aux revêtements des groupes réductifs quasi-déployés qui proviennent des extensions centrales de ces groupes par K_2 . On construit un L-groupe pour un tel revêtement, et on conjecture une paramétrisation de ses représentations irréductibles « spécifiques » (en anglais, « genuine ») par les paramètres de Langlands à valeurs dans ce L-groupe. En fait on donne deux constructions du L-groupe, qui sont reliées l'une à l'autre en fin d'article. La conjecture de Langlands locale proposée pour ces revêtements (LLCC) est prouvée pour les revêtements de tores déployés, les représentations sphériques dans le cas p-adique et les séries discrètes pour les revêtements doubles de groupes semi-simples réels. L'introduction du L-groupe permet de définir des fonctions L partielles et d'exprimer la fonctorialité, y compris le changement de base, pour ces représentations de revêtements.

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