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O-MINIMALITY AS AN APPROACH TO THE ANDRÉ-OORT CONJECTURE

by

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Abstract. – Employing a proof technique suggested by Zannier and first successfully implemented by Pila and Zannier to give a reproof of the Manin-Mumford conjecture on algebraic relations on torsion points of an abelian variety, Pila presented an unconditional proof of the André-Oort conjecture when the ambient Shimura variety is a product of modular curves. In subsequent works, these results have been extended to some higher dimensional Shimura and mixed Shimura varieties. With these notes we expose these methods paying special attention to the details of the Pila-Wilkie counting theorem.

Résumé **(O-minimalité comme approche à la conjecture d'André-Oort).** – En utilisant une technique de preuve, suggérée par Zannier et utilisée avec succès par Pila et Zannier, pour prouver la conjecture de Manin-Mumford sur les relations algébriques sur les points de torsion d'une variété abélienne, Pila a présenté une preuve inconditionnelle de la conjecture de André-Oort, lorsque la variété de Shimura ambiante est un produit de courbes modulaires. Ces résultats ont ensuite été étendus à d'autres variétés de Shimura et variétés de Shimura mixtes. Nous exposons ici ces méthodes, en accordant une attention particulière aux détails du théorème de comptage de Pila et Wilkie.

1. Introduction

In the paper **[57]** Pila gave the first unconditional proof of the André-Oort conjecture for mixed Shimura varieties expressible as products of curves. This fact on its own is a remarkable development, but the method of proof, coming as it does from the theory of o-minimality, constitutes a major breakthrough. Zannier had proposed that a theorem of Pila and Wilkie on counting rational points in definable sets in combination with suitable estimates on sizes of Galois orbits could be used to prove theorems

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in the vein of the André-Oort conjecture, and, indeed, in joint work with Pila [**63**] he implemented this strategy to reprove the Manin-Mumford conjecture. Subsequent work by many authors [**31, 46, 45, 44, 50, 59, 16**] has borne out the promise of this strategy and the pace of the continuing developments suggests that the project has not been played out.

These notes are based on a pair of lecture series I delivered in May 2011, one in Luminy to an assembled group of experts on the André-Oort conjecture with the aim of expositing the applications of the Pila-Wilkie counting theorem to diophantine geometric problems and then a second lecture series in Lyon to model theory students participating in a special Maloa (Mathematical Logic and its Applications) semester with the goal of explaining in detail the counting theorem itself. I have prepared two other accounts of these theorems [**71, 72**] to which the reader is referred for gentler introductions. In this paper, I will follow the proofs of the original papers fairly closely resisting the temptation to "simplify" those arguments. I do not claim any of the results explicated in this paper as my own, though, of course, any errors I may have inadvertently introduced are mine. The principal innovation is to have assembled in one place the key steps in these proofs.

The subject has progressed during the three years since the bulk of this paper was written. Most notably, Tsimerman has completed an unconditional proof of the André-Oort conjecture for $\mathscr{A}_g$, the coarse moduli space of principally polarized abelian varieties of dimension g, using the Pila-Zannier method [**79**]. The present text retains the structure and emphases of its 2012 version, but we conclude with a short section describing the current state of the art.

This paper is organized as follows. In Section 2 we outline the Pila-Zannier strategy. We follow with Section 3 in which we review the basic theory of o-minimality. In Section 4, the technical heart of this paper, we expose in detail the Pila-Wilkie counting theorem. Finally, in Section 5 we present some of the details of the proofs of the diophantine geometric theorems proven with these methods.

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2. Overview of the Pila-Zannier strategy

In this section we shall outline the main steps of the Pila-Zannier strategy for proving diophantine geometric theorems. Since the surveys [**71**] and [**72**] are devoted exactly to such outlines, we shall be brief here.

We are interested in proving theorems to the effect that if X is a "special" variety (a Shimura variety, an abelian variety, a moduli space for abelian varieties *et cetera*) and $Y \subseteq X$ is an irreducible closed subvariety containing a Zariski dense set of "special" points (special points in the sense of the theory of Shimura varieties, torsion point, CM-moduli points, *et cetera*), then Y is a "special" subvariety (variety of Hodge-type, group subvariety, submoduli variety, *et cetera*). In practice, we must specify the meaning of the term *special* (as we have suggested parenthetically). The Pila-Zannier strategy takes advantage of the theory of o-minimality which is essentially a theory of real geometry. As such, the technique applies only over the complex numbers, but one could speculate about extensions of the relevant counting theorems to analytic geometric situations over other local fields. Indeed, Comte, Cluckers and Loeser have formulated and proved a version of the Pila-Wilkie counting theorem for sets defined using p-adic analytic functions [**12**]. Subsequently, Chambert-Loir and Loeser have shown how to use this nonarchimedian counting theorem to prove functional transcendence results for maps coming from p-adic analytic uniformizations [**11**].

The first step in the Pila-Zannier strategy is to realize the complex algebraic variety $X(\mathbb{C})$ analytically as a coset space. That is, we seek some complex homogenous space $\mathfrak{X}$ for the action of some (open subgroup of a) real algebraic group $G(\mathbb{R})$ by analytic automorphisms so that some analytic function $\pi : \mathfrak{X} \to X(\mathbb{C})$ represents $X(\mathbb{C})$ as the quotient $\Gamma\backslash\mathfrak{X}$ where $\Gamma \leq G(\mathbb{R})$ is an arithmetic subgroup. For example, if X is an abelian variety over $\mathbb{C}$ of dimension g, then $X(\mathbb{C})$, being a complex torus, may be expressed as $\mathbb{C}^g/\Lambda$ for some lattice Λ. In this case, we would take $\mathfrak{X} = \mathbb{C}^g$ and $G = \mathbb{G}_a^{2g}$ acting via an appropriate real analytic trivialization of $\mathbb{C}^g$ as $\mathbb{R}^{2g}$ for which Λ is identified with $\mathbb{Z}^{2g}$. In the case that $X = \mathbb{A}^1$ regarded as the j-line, then we could take

$$\mathfrak{X} = \mathfrak{h} := \{\tau \in \mathbb{C} \ : \ \mathrm{Im}(\tau) > 0\}$$

to be the upper half plane, $G = \mathrm{PSL}_2$ to be the special linear group acting via fractional linear transformations, $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ and $\pi := j : \mathfrak{X} \to \mathbb{A}^1(\mathbb{C})$ to be the j-function. The reader might object that as the irreducible closed subvarieties of $\mathbb{A}^1$ are not particularly complicated, being either points or the full space, the problem considered here is trivial. Treat instead $X = \mathbb{A}^N$ regarded as the moduli space of products of N elliptic curves, $\mathfrak{X} = \mathfrak{h}^N$ and $\pi : \mathfrak{X} \to \mathbb{A}^N(\mathbb{C})$ given by

$$(\tau_1, \ldots, \tau_N) \mapsto (j(\tau_1), \ldots, j(\tau_N))$$

Of course, we need to be somewhat careful about how we choose the analytic covering $\pi : \mathfrak{X} \to X(\mathbb{C})$. In particular, we wish to have that the special points in $X(\mathbb{C})$ come from arithmetically simple points in $\mathfrak{X}$. What is meant by *arithmetically simple*? We shall ensure that $\mathfrak{X} \subseteq \mathbb{C}^M$ is an open subset of some complex affine space. Thus, it would make sense to ask whether a point in $\mathfrak{X}$ is rational or algebraic. In practice, we might like for the special points in $X(\mathbb{C})$ to be the images of the rational points in $\mathfrak{X}$, or possibly just algebraic points in $\mathfrak{X}$ of some bounded degree. With our example of X a complex abelian variety, the set of torsion points on $X(\mathbb{C})$ is exactly the image of $\mathbb{Q}^{2g}$ under the analytic covering map. In the case of the j-function giving a covering of $\mathbb{A}^1(\mathbb{C})$ by $\mathfrak{h}$, the set of special points, the j-invariants of elliptic curves with complex multiplication, is the image of the quadratic imaginary numbers. In the general applications of this method, we shall arrange that the set of preimages of special points under the covering map be the set of algebraic points in $\mathfrak{X}$ of degree at most d over $\mathbb{Q}$ for some fixed natural number d.

Once we have found the desired analytic covering map, the problem of describing the set of special points on the algebraic subvariety $Y \subseteq X$ may be converted to the problem of calculating the set of algebraic points of degree $\leq d$ on the analytic variety $\mathfrak{Y} := \pi^{-1}Y$. On the face of it, such a move converts a difficult problem to an intractable one as there is very little in general that one can say about the algebraic points on an analytic variety and the known theorems about the rational points on algebraic varieties are amongst the deepest in all of mathematics. To exploit this translation from special points on an algebraic variety to rational (or algebraic of bounded degree) points on an analytic variety we use the theory of definability in o-minimal structures. The covering map $\pi : \mathfrak{X} \to X(\mathbb{C})$ is almost never definable in a logically tame structure in any sense, but if we were to restrict π to an appropriate fundamental domain $\mathfrak{D} \subseteq \mathfrak{X}$ then the whole situation is often definable in an o-minimal expansion of the real numbers.

In the cases we have been considering, o-minimal definability takes on a very concrete form. Using real and imaginary parts we identify $\mathbb{C}$ with $\mathbb{R}^2$, and hence, $\mathbb{C}^N$ with $\mathbb{R}^{2N}$. By a semialgebraic set we mean a subset of $\mathbb{R}^{2N}$ defined by a finite boolean combination of conditions of the form $f(x_1, \ldots, x_{2n}) \geq 0$ where f is a polynomial with real coefficients. In the cases we have been considering, the fundamental domain $\mathfrak{D}$ may be taken to be semialgebraic. Indeed, when X is a complex abelian variety, then the natural choice for $\mathfrak{D}$ would be $[0,1)^{2g}$. In the case of the covering of the affine line by the j-function, the usual fundamental domain,

$$\mathfrak{D} := \{z \in \mathbb{C} \ : \ \frac{-1}{2} \leq \mathrm{Re}(z) < \frac{1}{2} \text{ and } |z| \geq 1\},$$

is easily seen to be semialgebraic. We say that a function is *restricted analytic* if it is the restriction of a real analytic function on some open set to a compact box. By an *explicitly definable* function we mean the restriction to a semialgebraic domain of a function built as a composition of polynomials, restricted analytic functions, and the real exponential function. The covering maps we have been considering are explicitly definable. In the case of the covering of an abelian variety $\pi : \mathbb{C}^g \to X(\mathbb{C})$ since π is globally analytic and the fundamental domain $\mathfrak{D}$ is contained in a compact box, one sees that the restriction of π to $\mathfrak{D}$ is already the restriction of a restricted analytic function to a semialgebraic set. For the j-function, one sees from the q-expansion of j, that the restriction of j to $\mathfrak{D}$ may be realized as the restriction to a semialgebraic set of the composite of a restricted analytic function with a function built from restricted analytic functions and the real exponential function.

At this point we may invoke the Pila-Wilkie counting theorem (or one of its refinements) to say something about the distribution of algebraic points on $\widetilde{\mathfrak{Y}} := \mathfrak{D} \cap \mathfrak{Y} = (\pi \restriction \mathfrak{D})^{-1}Y(\mathbb{C})$. The counting theorem says that after accounting for rational points which might lie on semialgebraic sets, there are subpolynomially many rational points on a definable set. Let us be a little more precise.