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THE CROSS SECTION MAP FOR GEODESIC FLOWS RELATED TO THE HECKE AND PICARD GROUPS

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Abstract. — We consider the Hecke group G_4 and the Picard group $\mathrm{PSL}(2,\mathbb{Z}[i])$. In the double of the quotient space of the upper half-space by $\mathrm{PSL}(2,\mathbb{Z}[i])$, we find a double cover of the quotient surface of the upper half-plane by G_4. We analyze the cross section map of the geodesic flow on this surface by using the graphic method of Adler-Flatto.

Résumé **(L'application de premier retour pour les flots géodésiques liés aux groupes de Hecke et de Picard)**

Nous considérons le groupe de Hecke G_4 et le groupe de Picard $\mathrm{PSL}(2,\mathbb{Z}[i])$. Dans le double de l'espace quotient du demi-espace supérieur par $\mathrm{PSL}(2,\mathbb{Z}[i])$, on trouve un double revêtement de la surface quotiente du demi-plan supérieur par G_4. Nous analysons l'application de premier retour du flot géodésique sur cette surface en utilisant la méthode graphique d'Adler-Flatto.

1. Introduction

The classical Markoff spectrum for $\mathbb{Q}$ has been studied from various points of view: continued fractions, quadratic forms and geometry. In [1], we gave a geometric interpretation of the Markoff spectrum for $\mathbb{Q}(i)$, generalizing the geometric study of the Markoff spectrum for $\mathbb{Q}$. A point we clarified is that the Picard group $\mathrm{PSL}(2,\mathbb{Z}[i])$ naturally contains the Hecke group of order 4, G_4 (see §2), and that this subgroup captures the discrete part of the Markoff spectrum for $\mathbb{Q}(i)$.

In the present paper, we discuss the coding of the geodesic flow on a special surface in the double of the quotient space of the upper half-space $\mathbb{H}^3$ by $\mathrm{PSL}(2,\mathbb{Z}[i])$. The special surface can be identified with a double cover of the quotient surface of the upper half-plane $\mathbb{H}^2$ by G_4. We show that the codings of the geodesic flows on the

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special surface and on $\mathbb{H}^2/G_4$ are characterized in a similar way (see §4). Thus, we find another 'good' feature of the Picard group depending on its subgroup G_4.

To analyze a cross section map of the geodesic flow on the modular surface, R. Adler and L. Flatto used a graphic method in [2]. Let us summarize their method. The geodesic flow on the modular surface is coded by using the endpoints of the geodesic, which are denoted by η and ξ. Since the universal cover of the modular surface is the upper half-plane $\mathbb{H}^2$ and its boundary is the real line, we have $(\eta, \xi) \in (\mathbb{R} \cup \{\infty\})^2$ and the geodesic is represented by the (η, ξ)-coordinate. Thus, the cross section map (first return map) can be simply expressed in the (η, ξ)-plane, where the cross section is the set of outward unit vectors whose base points are on the boundary of the usual fundamental domain of the modular group. Because of the shape of the domain, it is called the *curvilinear map*. Even if this coding is natural, it does not have a Markovian partition. By a simple geometrical recoding, the *rectilinear map* is obtained from the curvilinear map. Its domain is composed of rectilinear regions. The vertical and horizontal directions are contracting and expanding, respectively, under the rectilinear map. That is, the rectilinear map has a Markovian partition. Moreover, there is a conjugacy map between the curvilinear and the rectilinear maps which is the identity on most of its domain.

We apply Adler and Flatto's method to analyze a cross section map of the geodesic flow on our surface, *i.e.*, the immersion of the double cover of $\mathbb{H}^2/G_4$ in the double of $\mathbb{H}^3/\mathrm{PSL}(2, \mathbb{Z}[i])$. Note that $\mathbb{H}^2/G_4$ is one of the natural generalizations of the modular surface. We proceed as follows: in §3 we express the Poincaré return map of a cross section of the geodesic flow on our surface by using the endpoints of the geodesics (to define a curvilinear map T_C); in §4 we construct its linearized version (to define a rectilinear map T_R), and find a conjugacy map Φ between them. The conjugacy map Φ is the identity on most of the set on which it is defined. (The same situation arises with the geodesic flow on the modular surface.) This fact is stated in Theorem 4.1, which is our main result. Note that the cross section map we obtain corresponds to a continued fraction expansion of complex numbers whose partial quotients are of the form $k(1 + i)$ and $k(1 - i)$, $k \in \mathbb{Z}$.

In the construction of the conjugacy map, we clarify the relations between some of the generators of the Picard group. The graphs (see Figures 2 and 3) used in this paper are more complicated than the corresponding ones for the modular surface (see [2]). That means that a Markovian partition of our geodesic flow is more complicated than the one of the geodesic flow in the modular surface. An interest of the graphic method is that it makes the difference intelligible.

For the geodesic flow in the quotient surface $\mathbb{H}^2/G_4$, it is also possible to construct the curvilinear map, the rectilinear map and the conjugacy map between them. The construction of these maps is almost the same, as we produce in §3 and §4. We only write down these maps in a remark after Theorem 4.1. Note that these maps and their graphs are almost the same as the ones for the geodesic flow on the special surface in the 3-manifold.

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2. Hecke group, Picard group

We begin by introducing the groups and spaces which we will use in this paper. We always identify a point (x, y) in $\mathbb{R}^2$ with $x + yi$ and a point (x, y, t) in $\mathbb{R}^3$ with $x + yi + tj$, where $i^2 = j^2 = -1$. The upper half-plane $\{z = x + iy \in \mathbb{C} \mid y > 0\}$ is denoted by $\mathbb{H}^2$ and the upper half-space $\{z + jt \mid z \in \mathbb{C},\ t > 0\}$ is denoted by $\mathbb{H}^3$. Suppose that they are endowed with the hyperbolic metrics $ds^2 = (dx^2 + dy^2)/y^2$ and $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$, respectively.

The Hecke group G_4 is generated by two elements (see [5])

$$G_4 := \left\langle A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ P := \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \right\rangle.$$

The Picard group Γ is generated by A, T, U and L. It is denoted by $\Gamma = \langle A, T, U, L \rangle$, where

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U := \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad L := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The Picard group is also represented by $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[i])$, where $\mathbb{Z}[i]$ is the set of complex integers (see [4]). The group G_4 acts on $\mathbb{H}^2$ by fractional linear transformations. The group Γ acts on $\mathbb{C}$ by fractional linear transformations and on $\mathbb{H}^3$ by their Poincaré extensions (see [3]). In what follows we always identify an element $g \in G_4$ with the fractional linear transformation and an element $g \in \Gamma$ with the fractional linear transformation or its Poincaré extension induced by g.

The Hecke group G_4 acts on $\mathbb{H}^2$ discontinuously and a fundamental domain of G_4 can be represented as follows:

$$F_4 := \left\{ x + iy \in \mathbb{H}^2 \mid x^2 + y^2 > 1,\ |x| < \frac{1}{\sqrt{2}} \right\},$$

where $|x|$ denotes the absolute value of x. Topologically this is the same as the modular surface, that is, a sphere minus a point. There are two singular points on the quotient surface F_4: i and $(-1+i)/\sqrt{2}$, this latter is identified with $(1+i)/\sqrt{2}$. Their ramification numbers are 2 and 4, respectively, which come from $A^2 = (AP)^4 = Id$. The latter is different from the singular point on the modular surface coming from $(AT)^3 = Id$.

The Picard group Γ acts on $\mathbb{H}^3$ discontinuously and a fundamental region of Γ can be represented as follows:

$$F := \left\{ x + iy + jt \in \mathbb{H}^3 \mid x^2 + y^2 + t^2 > 1,\ |x| < \frac{1}{2},\ 0 < y < \frac{1}{2} \right\}.$$

The region F has a single parabolic vertex at ∞ and has a finite volume. (See Example 15 in [6].) Then $\hat{F} = F \cup LF$ is a fundamental region of $\langle A, T, U \rangle$. We call $\hat{F}$ the *fundamental polyhedron*.

Define lines l_1, l_2 on $\mathbb{C}$ as $l_1 := \{(x, y, 0) \mid x = y\}$, $l_2 := \{(x, y, 0) \mid x = -y\}$ and define planes $\hat{W}_1, \hat{W}_2$ in $\mathbb{H}^3$ as $\hat{W}_1 := \{(x, y, t) \mid x = y,\ t > 0\}$, $\hat{W}_2 := \{(x, y, t) \mid x = -y,\ t > 0\}$. Note that l_1, l_2 are boundaries of $\hat{W}_1, \hat{W}_2$, respectively. Take the following two matrices:

$$M_1 := \begin{pmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 := \begin{pmatrix} \frac{1}{\sqrt{2}}(1-i) & 0 \\ 0 & 1 \end{pmatrix}.$$

They give identifications between the planes $\hat{W}_1$, $\hat{W}_2$ and the upper half-plane $\mathbb{H}^2$. Indeed,

$$\mathbb{H}^2 \ni x + it \longleftrightarrow M_1(x + jt) = \frac{1}{\sqrt{2}}(1+i)x + jt \in \hat{W}_1,$$
$$\mathbb{H}^2 \ni x + it \longleftrightarrow M_2(x + jt) = \frac{1}{\sqrt{2}}(1-i)x + jt \in \hat{W}_2.$$

Under this identification, if we use the coordinates $x + it$, $\hat{W}_1$ and $\hat{W}_2$ are denoted simply by W_1 and W_2, respectively. Define $\hat{k} = k+1 \pmod 2$, $\hat{k} \in \{1, 2\}$ for $k \in \{1, 2\}$.

We consider the action on $\hat{W}_k$ defined by the following matrices:

$$P_1^{\pm} = \pm TU = \begin{pmatrix} 1 & \pm(1+i) \\ 0 & 1 \end{pmatrix}, \quad P_2^{\pm} = \pm TU^{-1} = \begin{pmatrix} 1 & \pm(1-i) \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.1

(i) *The action of A switches the planes $\hat{W}_1$ and $\hat{W}_2$, that is, $A\hat{W}_k = \hat{W}_{\hat{k}}$.*
(ii) *The action $P_k^{\pm}$ is a parallel displacement on $\hat{W}_k$ and satisfies $P_k^{\pm}\hat{W}_k = \hat{W}_k$.*
(iii) *The action $P_k^{\pm}$ on $\hat{W}_k$ is equivalent to the action P on W_k.*

Proof. — The assertions (i) and (ii) are easily checked by a calculation. For (iii) we can easily check that for $x + it \in \mathbb{H}^2$, $P(x + it)$ is identified with $P_k^{+} M_k(x + jt)$. □

The intersection of $\hat{W}_1 \cup \hat{W}_2$ with the fundamental polyhedron $\hat{F}$ can be identified with two sheets of the fundamental domain F_4, that is, $F_{41} \cup F_{42}$, where F_{4k} denotes the domain F_4 on W_k. Moreover, $F_{41} \cap F_{42}$ and the boundary of $F_{41} \cup F_{42}$ are the lines from the singular points to ∞, *i.e.*, the vertical lines which lie over $0 \in \mathbb{C}$ and $(1 + i)/2$. The point $(1 + i)/2$ is identified with $(-1 + i)/2$, $(-1 - i)/2$ and $(1 - i)/2$ by the action of T and U.