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E. H. El Abdalaoui & M. El Machkouri & A. Nogueira

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A CRITERION OF WEAK MIXING PROPERTY

by

E. H. El Abdalaoui, M. El Machkouri & A. Nogueira

Abstract. — We give a new criterion of the weak mixing property of a dynamical system in Ergodic Theory. The criterion is related to a conjecture due to Sinai.

***Résumé* (Sur un critère pour la propriété de mélange faible).** — Nous donnons un critère pour la propriété de mélange faible d'un système dynamique en théorie ergodique. Notre critère est en relation avec une conjecture de Sinaï.

1. Introduction

The purpose of this article is to present a criterion of weak mixing property in Ergodic Theory. First we shall recall some well-known weak mixing criteria. It turns out that our criterion is related to a conjecture of Sinai [10] regarding the connection between the behavior of the sequence of operators $S_n = \sum_{i=0}^{k-1} T^i$ on L^2 and the mixing property. Our principal motivation came for Zorich theorem [12] and the recent work of Marmi-Moussa and Yoccoz in the context of interval exchange maps. In this class Avila and Forni obtain recently that the weak mixing holds almost surely [1]. They use the methods developed from Zorich [12] and Marmi-Moussa-Yoccoz [11] combined with the recent work of Forni [4]. It is our hope that our criterion can be used to obtain in this context a simple proof. We may motivate our work also by the work of Delarue-Velenik and Janvresse [7] on the deviation of ergodic sum in the context of Pascal-adic Transformations. For these transformations, whether weak mixing is satisfied is an open question. It is our hope that our criterion can be used to solve this question.

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Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system, where $(\Omega, \mathcal{F}, \mu)$ is a probability space and $T : \Omega \to \Omega$ is an automorphism which preserves the measure μ (*i.e.*, $\mu(T^{-1}A) = \mu(A)$, for all $A \in \mathcal{F}$). We say that a measurable set A is invariant under T, if $\mu(T^{-1}A\Delta A) = 0$. We say that T is ergodic, if every measurable set invariant under T has measure 0 or 1. For every measurable function f and for all $n \geq 1$, we set

$$S_n(f) = \sum_{i=0}^{n-1} f \circ T^i.$$

Then T is mixing, if for every $f \in L^2$ with zero mean, we have

$$\lim_{n \to +\infty} \left| \int f \circ T^n . \overline{f} \, d\mu \right| = 0.$$

We say that the transformation T is weak mixing, if for every $f \in L^2$ with zero mean, we have

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \int f \circ T^n . \overline{f} \, d\mu \right| = 0.$$

A classical characterization of the weak mixing property is that 1 is the only eigenvalue of the operator $U_T : f \longmapsto f \circ T$ and the eigenvalue is simple. Halmos and Rokhlin in [6] proved by category argument, the existence of maps on the interval preserving Lebesgue measure which are weak mixing, but not mixing. Later, Chacon [2, 3] and Katok-Stepin [9] give explicit examples of such maps.

For Chacon's examples, one can show in the first case [3] that there exists a sequence of integers h_k such that the sequence $U_T^{h_k}$ converges weakly to $\frac{1}{2}(I + U_T)$ and in the second case [2] that the limit is $\sum_{n \in \mathbb{N}} \frac{U_T^n}{2^n}$. In [9] it is proved that there exists a sequence of integers n_k such that $U_T^{n_k}$ converges weakly to $(1 - \alpha)U_T + \alpha \mathbb{P}_c$, where $0 < \alpha < 1$, and $\mathbb{P}_c$ is the projection on the constants.

Next we unify these two conditions.

Proposition. — *Let T be an ergodic dynamical system. Assume that there exist non-negative numbers $(a_i)_{i \in \mathbb{Z}}$ such that $\sum_{i \in \mathbb{Z}} a_i = 1$ and there exist i such that $a_i a_{i+1} \neq 0$. Assume that*

$$(1) \qquad (1 - \alpha) \sum_{i \in \mathbb{Z}} a_i U_T^i + \alpha \mathbb{P}_c$$

belongs to the weak closure of $\{U_T^n : n \in \mathbb{Z}\}$, where $\alpha \in [0, 1)$. Then T is weak mixing, but not mixing.

Proof. — It is easy to see that, if there exist a sequence (n_k) and an integer i_0 such that $a_{i_0} > 0$ and for every measurable set A, we have

$$\lim_{k \longrightarrow +\infty} \mu(T^{n_k} A \cap A) \geq (1 - \alpha) a_{i_0} \mu(A),$$

then T is not mixing. Now we shall prove that the above conditions imply that T is weak mixing. The case $\alpha > 0$ is easy, we shall assume that $\alpha = 0$. Let λ be an eigenvalue of U_T and choose the correspondent eigenfunction f such that $\int f\,d\mu = 0$ and $\|f\|_2 = 1$. According to (1), there exists a sequence of integers (n_i) such that λ^{n_i} converge to $P(\lambda) \stackrel{\text{def}}{=} (1-\alpha)\sum_{i\in\mathbb{Z}} a_i\lambda^i$, it follows that $|P(\lambda)| = 1 = \left|\int \lambda^n d\sigma\right|$, where σ is the probability measure on $\mathbb{Z}$ defined by

$$\sigma\{i\} = a_i, \forall i \in \mathbb{Z}.$$

So we have equality in the inequality of Cauchy-Schwarz. We deduce that λ^n is constant for almost all n with respect to σ, *i.e.*, $\lambda = 1$, since there are two identical consecutive powers of λ . □

We recall that the two weak mixing maps given in [**2, 3**] are prime, so without mixing factor. For them there exists a sequence of integers n_k such that $U_T^{n_k}$ converges weakly. We note that Chacon argument is in the context of rank one transformations. Rank one transformations are ergodic and for ergodic transformations we have the following criterion due to Halasz.

***Theorem* (Halasz [5], 1976).** — *Let $(\Omega, \mathcal{F}, \mu, T)$ be an ergodic dynamical system. Assume that there exists a set $A \in \mathcal{F}$ with positive measure such that for all $n \geq 1$, $|S_n(\mathbb{1}_A - \mu(A))|$ is bounded with positive probability. Then $e^{2i\pi\mu(A)}$ is an eigenvalue of T, that is, there exists a measurable function $f \neq 0$ such that*

$$f \circ T = e^{2i\pi\mu(A)} f.$$

Conversely, if there exists μ_0 in $(0,1)$ such that $e^{2i\pi\mu_0}$ is an eigenvalue of T, then there exist a set $A \in \mathcal{F}$ satisfying $\mu(A) = \mu_0$ and a constant $C > 0$ such that for all $n \geq 1$

$$|S_n(\mathbb{1}_A - \mu(A))| \leq C \quad a.s.$$

Proof. — For any measurable set A, put

$$f_A = \chi_A - \mu(A).$$

Observe that the set $\{x \ : \ |S_n(\chi_A - \mu(A))| \ \text{bounded}\ \}$, is an invariant set, in fact, we have the following relation

$$S_n(f_A) \circ T = S_n(f_A) - f_A + f_A \circ T^n.$$

It follows, from the ergodicity, that this set is full measure set. Now, put, for almost all x,

$$h(x) = \inf_{n\geq 1} S_n(f_A)(x),$$

and observe that we have the following

$$h \circ T(x) \geq h(x) - f_A(x).$$

We get that $g(x) = f_A(x) + h \circ T(x) - h(x)$ is a positive function. Apply the pointwise ergodic theorem to obtain, for almost every x,

$$\frac{1}{n}\sum_{k=1}^{n} g \circ T^k(x) \xrightarrow[n\to\infty]{} \int g\, d\mu.$$

But, by the definition of g and the pointwise ergodic theorem, we have, for almost every x,

$$\frac{1}{n}\sum_{k=1}^{n} g \circ T^k(x) = \frac{1}{n}\sum_{k=1}^{n} f_A \circ T^k(x) + \frac{1}{n}(h(T^n x) - h(x)) \xrightarrow[n\to\infty]{} 0$$

since $\int f_A = 0$. Now, by the Poincaré recurrence theorem, there exist infinitely many integers n for which $T^n(x)$ is in the set $\{|h| \leq M\}$ for some $M > 0$. It follows that

$$f_A = h - h \circ T.$$

Put, now

$$H(x) = e^{2\pi i h(x)},$$

and observe that H is a eigenfunction of T. Conversely, Let H be any eigenfunction. Then, by the ergodicity, we may assume that $H(x) \neq 0$, for almost all x. Define h by

$$h(x) = \frac{1}{2\pi}\mathrm{Arg}(H(x)).$$

It follows that

$$h \circ T(x) = h(x) + \mu_0 \mod 1.$$

This implies that $h(T(x)) - h(x) \in \{\mu_0, \mu_0 - 1\}$, *i.e.*, $h(T(x)) - h(x) + \mu_0$ is any indicator function and the result follows and the proof of the theorem is complete. □

2. Main result

In this section we shall state the main result of this note. Let us point out that this result is connected to the following conjecture due to Kachurovskii [8].

Conjecture. — *The map T is mixing if, and only if, $\|S_n(f_A)\|_2 \to \infty$, as $n \to \infty$, for all positive measure sets A.*

The fact that the condition is necessary was conjectured by Sinai and established by Leonov [10]. It is an open question whether the converse is true. We are not able here to give the positive answer to this question but we shall give the positive answer to the analogue of this conjecture in the case of weak mixing, more precisely we have the following.