Séminaires & Congrès

COLLECTION SMF

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ÉCOLE DE THÉORIE ERGODIQUE

Numéro 20 Y. Lacroix, P. Liardet, J.-P. Thouvenot, éds.

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
20, 2010, p. 147–161

TOEPLITZ SUBSHIFTS AND ODOMETERS FOR RESIDUALLY FINITE GROUPS

by

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Abstract. — Let G be a countable group and A be a finite set. In this note we present a class of minimal subshifts of A^G generalizing Toeplitz subshifts for $G = \mathbf{Z}$. We state some of their properties and in particular we describe their maximal equicontinuous factor.

Résumé **(Sous-décalages de Toeplitz et odomètres sur les groupes résiduellement finis)**

Soit G un groupe dénombrable et A un ensemble fini. Dans cette note on présente une classe de sous-décalages minimaux de A^G généralisant les sous-décalages de Toeplitz pour $G = \mathbf{Z}$. Après avoir donné quelques propriétés de ces systèmes, nous décrivons leur facteur équicontinu maximal.

1. Introduction

Let G be a countable group with unit element 1_G. A *G-space* is a couple (G, X) where X is a compact metrizable space together with a continuous action of G. Let A be a finite set of cardinality $|A|$ with the discrete topology. Let $A^G = \{x = (x_g)_{g \in G} \colon x_g \in A\}$ the set of all functions from G to A endowed with the product topology. The *full G-shift* on A^G is the G-space (G, A^G) where the action of G is defined by

$$(gx)_{g'} = x_{g'g}$$

for all $x \in A^G$ and $g, g' \in G$. A closed G-invariant subset $Y \subset A^G$ is called a $subshift$.

An element $x \in A^G$ is a *Toeplitz element* if for all $g \in G$ there is a subgroup H of finite index in G such that the restriction of x to the left coset gH is a constant function. A *Toeplitz subshift* is an orbit closure in A^G of a Toeplitz element. They are minimal systems (Corollary 3.4).

2000 Mathematics Subject Classification. — 37B05, 37B10.
Key words and phrases. — Toeplitz subshift, odometer.

Toeplitz subshifts over $\mathbf{Z}$ were introduced in 1969 by Jacobs and Keane [12] and motivated by a construction of O. Toeplitz [18] of an almost periodic function on the real line. A sequence $x \in A^{\mathbf{Z}}$ is a Toeplitz element if and only if the set of integers can be decomposed into arithmetic progressions such that x is constant on each progression. The class of Toeplitz subshifts in $A^{\mathbf{Z}}$ have been intensively studied for their topological and ergodic properties (see for instance [16], [19], [5], [10]).

Toeplitz subshifts have been generalized to the groups $\mathbf{Z}^n$ and to more general groups independently in [2], [3], [4] and [13]. Many results for the $\mathbf{Z}$-Toeplitz can be generalized to more general group actions and the essential concepts seem to become more transparent.

Every G-space has a maximal equicontinuous factor, unique up to conjugacy (see Section 2 for the definition). For $G = \mathbf{Z}$, the maximal equicontinuous factor of a Toeplitz subshift is an odometer (sometimes called an adding machine) [8]. In fact, Toeplitz subshifts in $A^{\mathbf{Z}}$ are exactly the minimal subshifts which are almost 1-1 extensions of odometers ([15], [14], see also [7, Theorem 6]). The notion of odometer can be generalized for arbitrary groups (Section 5) and the preceding characterization result is still true for G-Toeplitz subshifts (Theorem 5.8).

Let us point out that the main statements in this note are extensions to G-actions of results about $\mathbf{Z}$-Toeplitz by using techniques and notions contained in [6] and [19]. These generalizations were also done by M.I. Cortez and S. Petite in [4] (see also [3] for $\mathbf{Z}^d$-Toeplitz), but the approach adopted here is slightly different.

The paper is organized as follows. In Section 2, we recall some general notions about topological dynamics. Toeplitz subshifts and related notions are defined in Section 3. Some convenient lemmas used throughout this paper are also given in this section. Part 4 is devoted to constructions of Toeplitz subshifts. We show that if the group G contains subgroups of arbitrary large index (*e.g.*, infinite residually finite groups), then one can always construct infinite Toeplitz subshifts of A^G (Theorem 4.1). In Section 5, we generalize the notion of odometer to arbitrary groups and we describe the maximal equicontinuous system of a G-Toeplitz subshift.

The author would like to thank the C.I.R.M. (Luminy, Marseille) for their hospitality during the Ergodic Theory School April 24 – 28, 2006.

2. Background on topological dynamics

Let (G, X), (G, Y) be G-spaces. A map $f\colon X \to Y$ is said to be *G-equivariant* if f commutes with the action, *i.e.*, if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. Let $f\colon X \to Y$ be a continuous G-equivariant map. If f is surjective then (G, Y) is called a *factor* of (G, X) and f a *factor map*. In this case (G, X) is said to be an *extension* of (G, Y). If f is a homeomorphism then (G, X) and (G, Y) are said to be *topologically conjugate.*

A factor map $f\colon X \to Y$ is an *almost 1-1* map if the set of *singleton fibers* $Y_1 = \{y \in Y\colon |f^{-1}(y)| = 1\}$ (which is in fact a G_δ *set, i.e.*, a countable intersection of open sets) is dense in Y. The G-space (G, X) (resp. (G, Y)) is then called an *almost 1-1-extension of* (G, Y) (resp. *almost 1-1-factor* of (G, X)).

Let d be a metric on X compatible with the topology. The G-space (G, X) is said to be *equicontinuous* if for all $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(gx, gy) < \epsilon$ for all $g \in G$. Observe that this notion doesn't depend on the choice of the compatible metric d since X is compact. It can be proved (see [9] or [1]) that every G-space (G, X) has an unique (up to conjugacy) *maximal equicontinuous factor* (G, Y), *i.e.*, a factor map $f\colon X \to Y$ such that for every factor map $h\colon X \to Z$, where (G, Z) is an equicontinuous G-space, there is a factor map $f'\colon Y \to Z$ such that $h = f' \circ f$.

A non empty G-space (G, X) is called *minimal* if all its orbits are dense in X, *i.e.*, $\overline{Gx} = X$ for all $x \in X$. It is a consequence of Zorn's Lemma that every G-space contains a minimal G-invariant subset ([1, Theorem I.4]).

A subset $S \subset G$ is called *syndetic* if there exists a finite set $F \subset G$ such that $G = FS$. For instance a subgroup of G is syndetic if and only if it is of finite index.

An element $x \in X$ is called *almost periodic* if for any neighborhood V of x there exists a syndetic set $S \subset G$ such that $Sx \subset V$.

The following Theorem is a well known characterization of minimal G-spaces:

Theorem 2.1. — *Let (G, X) be a G-space. Then $x \in X$ is an almost periodic point if and only if $\overline{Gx}$ is a minimal G-space.*

Proof. — See for instance [1, Theorem I.7, page 11]. □

3. Toeplitz subshifts

3.1. Periodic parts. — Let $x \in A^G$ and H be a subgroup of G. The *H-periodic part* $\mathrm{Per}_H(x) \subset G$ of x is defined by

$$\mathrm{Per}_H(x) = \{g \in G\colon x_{gh} = x_g \quad \text{for all } h \in H\}.$$

In other words $\mathrm{Per}_H(x)$ is the union of the left cosets gH of H such that the restriction of x to gH is a constant function. If K and H are subgroups of G with $K \subset H$ then $\mathrm{Per}_H(x) \subset \mathrm{Per}_K(x)$. Define also for all $a \in A$

$$\mathrm{Per}_H(x, a) = \{g \in G\colon x_{gh} = a \quad \text{for all } h \in H\}$$

and observe that we have the following partition of $\mathrm{Per}_H(x)$:

$$\mathrm{Per}_H(x) = \bigsqcup_{a \in A} \mathrm{Per}_H(x,a).$$

Lemma 3.1. — *Let $x \in A^G$ and H be a subgroup of G. Then one has*

$$\mathrm{Per}_H(gx,a) = \mathrm{Per}_{g^{-1}Hg}(x,a)g^{-1}$$

for all $g \in G$ and $a \in A$.

Proof. — Fix $g \in G$. The Lemma follows from the equality

$$(gx)_{g'h} = x_{g'gg^{-1}hg} \quad \text{for all } g' \in G \text{ and } h \in H. \quad \square$$

3.2. Toeplitz elements. — An element $x \in A^G$ is called a *Toeplitz element* if for all $g \in G$, there is a subgroup of finite index $H \subset G$ such that $g \in \mathrm{Per}_H(x)$.

Since G is a countable group, one has the following equivalent definition:

Lemma 3.2. — *An element $x \in A^G$ is a* Toeplitz element *if and only if there exists a sequence $H_0 \supset H_1 \supset \dots$ of subgroups of finite index in G such that $G = \bigcup_{n\in\mathbf{N}} \mathrm{Per}_{H_n}(x)$.*

Proof. — Let $x \in A^G$ be a Toeplitz element. Since G is countable, we can write $G = \{g_n : n \in \mathbf{N}\}$. We will construct the sequence (H_n) by induction. As x is a Toeplitz element, there exists a subgroup of finite index H_0 of G such that $g_0 \in \mathrm{Per}_{H_0}(x)$. Suppose that $H_0 \supset \cdots \supset H_n$ are constructed with the property that $g_k \in \mathrm{Per}_{H_k}(x)$ for $k = 0,\dots,n$. Let us define H_{n+1}. Since x is a Toeplitz element, there is a subgroup H of finite index in G such that $g_{n+1} \in \mathrm{Per}_H(x)$. Define the (finite indexed) subgroup $H_{n+1} = H \cap H_n$ of G. Then $g_{n+1} \in \mathrm{Per}_H(x) \subset \mathrm{Per}_{H_{n+1}}(x)$. It follows that $G = \bigcup_n \mathrm{Per}_{H_n}(x)$. $\square$

Let $x \in A^G$ be a Toeplitz element. If $H_0 \supset H_1 \supset \dots$ is a sequence of subgroups of finite index in G such that $G = \bigcup_{n\in\mathbf{N}} \mathrm{Per}_{H_n}(x)$, then one says that x is a Toeplitz element *with respect to* the sequence (H_n).

An element $x \in A^G$ is said to be *periodic* if its G-orbit is finite, or equivalently, if the *stabilizer* $\mathrm{Stab}(x)$ of x is of finite index in G. If x is periodic then $G = \mathrm{Per}_H(x)$, where $H = \mathrm{Stab}(x)$. Periodic elements are trivial examples of Toeplitz elements, but under certain assumptions, there exist non periodic Toeplitz elements (see Theorem 4.1, see also [13, Théorème 1.1]).

For $F \subset G$, denote by $\pi_F : A^G \to A^F$ the projection onto A^F.

Proposition 3.3. — *Let $x \in A^G$ be a Toeplitz element. Then x is almost periodic.*