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INFINITE CONVOLUTION OF BERNOULLI MEASURES, PV NUMBERS AND RELATED PROBLEMS IN THE DYNAMICS OF FRACTAL GEOMETRY

by

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Abstract. — The case of equality between the Minkowski and Hausdorff dimension for the graphs of Weierstrass-like functions, remains largely mysterious. However significant progresses have been obtained when the graph is self-affine: for instance the analysis of the limit Rademacher function by Przytycki & Urbański shows how this question is concerned with the arithmetics of the Pisot-Vijayaraghavan (PV) numbers and an *Erdős problem* about the so-called Infinite Convolution Bernoulli Measures (ICBMs). A related question is to understand the fine multifractal/dynamical structures of the special ICBM associated with the golden number and called the *Erdős measure*: from this point of view, we answer a question of Sidorov and Vershik about the Gibbs nature of the *invariant Erdős measure.*

***Résumé* (Problèmes de dynamique liés à la géométrie fractale de certaines convolution d'une infinité de mesures de Bernoulli en base de Pisot-Viraraghavan)**
Le cas d'égalité entre la dimension de Minkowski et la dimension de Hausdorff du graphe des fonctions *à la* Weierstrass, demeure toujours mystérieux. Cependant, des progrès significatifs ont été réalisés pour certains graphes autoaffines: par exemple, pour les fonctions de Rademacher, Przytycki & Urbański, montrent comment cette question est liée à l'arithmétique des nombres de Pisot-Vijayaraghavan (PV): on retrouve alors un *problème d'Erdős* sur certaines convolutions d'une infinité de mesures de Bernoulli (convolutions de Bernoulli). Une question attenante est de comprendre la structure multifractale de la *mesure d'Erdős i.e.* la convolution de Bernoulli associée au nombre d'or: de ce point de vue, nous répondons à une question posée par Sidorov et Vershik à propos de la nature gibbsienne de la *mesure d'Erdős* invariante.

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Introduction

Let $\alpha, \beta > 1$ be two real numbers with $\alpha/\beta > 1$ and note Γ the graph of the celebrated Weierstrass function (1872), $W : [0\,;1] \to \mathbf{R}$ such that

$$W(x) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \cos(2\pi\alpha^n x)\ ;$$

The function W is a classical example of a continuous nowhere differentiable map, which makes its graph Γ an interesting object of Fractal Geometry. We fix $\alpha = 2$ and include in our presentation the class of the Weierstrass-like functions $\mathcal{W}_\beta : [0\,;1] \to \mathbf{R}$ such that

$$\mathcal{W}_\beta(x) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \varphi(2^n x),$$

where $1 < \beta < 2$ and $\varphi : \mathbf{R} \to \mathbf{R}$ is a $\mathbf{Z}$-periodic function: in what follows Γ_β stands for the graph of $\mathcal{W}_\beta$. When φ is Lipschitz continuous, $\mathcal{W}_\beta$ turns out to be Hőlder continuous of exponent $\log\beta/\log 2$; the Hőlder exponent is then directly related to a classical upper-bound of the Minkowski dimension (*box counting dimension*), that is $\dim_M \Gamma_\beta \leq 2 - \log\beta/\log 2$. However, even if a Weierstrass-like function need not be continuous, the Minkowski dimension provides a general upper-bound of the Hausdorff dimension, so that [**29**, **37**]:

$$\dim_H \Gamma_\beta \leq 2 - \frac{\log\beta}{\log 2}. \tag{1}$$

The case of equality in (1) is a delicate problem still open for the Weierstrass functions. Among partial results for the Weierstrass-like functions, two significant progresses are concerned with self-affine graphs, the first one by Przytycki & Urbański [**37**] for the limit Rademacher functions, the second one by Ledrappier [**25**] for the Takagi functions. In both cases the problematic equality in (1) is proved to be related to an *Erdős problem* [**2**, **5**, **11**, **16**, **18**, **43**] about the family of probability measures ν_β $(1 < \beta < 2)$, and called Infinite Convolution of Bernoulli Measures (ICBMs): in Section 1, we focus our attention on the limit Rademacher functions to see how the arithmetic of β is involved and more precisely how PV-numbers yields strict inequality in (1).

The multifractal analysis of the measures ν_β is supposed to give precious informations about the fractal properties of ν_β and Γ_β; however, the task is difficult and only few results have been established. Section 2 is devoted to the statement of the multifractal formalism satisfied by the *Erdős measure*, that is $\nu = \nu_\beta$ for $\beta = (1+\sqrt{5})/2$ [**10**, **23**, **26**] (we refer to [**7**, **9**, **31**] for multifractal analysis of ν_β, for other PV-numbers β).

Following the seminal work of Alexander & Yorke [**2**] about the so-called *Fat Baker's Transformation*, a detailed analysis of dynamical/ergodic properties of the Erdős measure is given by Sidorov & Vershik [**42**]; in particular, it is proved

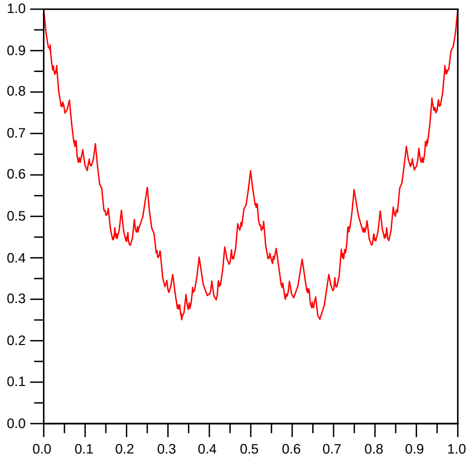

FIGURE 1. Graph Γ of the Weierstrass function W for $\alpha = 2$ and $\beta = (1+\sqrt{5})/2$: here $\dim_M \Gamma = 2 - \log\beta/\log 2$. The exact value of the Hausdorff dimension $\dim_H \Gamma$ is still unknown.

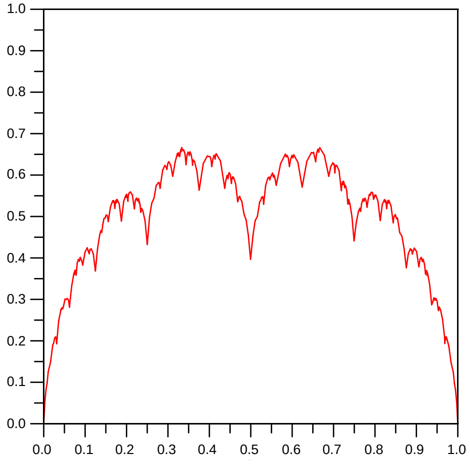

FIGURE 2. Graph of the Takagi function $\mathcal{W}_\beta$ with $\beta = (1+\sqrt{5})/2$ and $\varphi(x) = 1 - \mathrm{dist}(x, \mathbf{Z})$. By a result of Ledrappier [25], $\dim_M \Gamma_\beta = \dim_H \Gamma_\beta$ when the ICBM ν_β has Hausdorff dimension 1, which [43] arises for almost all β.

[42, Corollary 1.9] that there exists a unique probability measure ν' invariant w.r.t. multiplication by β (mod 1) and equivalent to ν. In Section 3 we present the result of Sidorov & Vershik about the measure ν', leading to the statement of our main result, Theorem 4.2, which answers a question in [42, Remark p. 222] about the Gibbs nature of ν'; the proof of Theorem 4.2, given in Section 4, makes use of Kac Recurrence Theorem, Abramov Formula and Thermodynamic Formalism for infinite alphabet shift spaces [28, 40, 44].

1. Limit Rademacher functions and ICBMs

Consider for any $i \in \{\mathbf{0}, \mathbf{1}\}$, the affine contraction $A_i : \mathbf{R}^2 \to \mathbf{R}^2$ such that $A_i(x,y) := A(x,y) + iV$, where A and V are identified to their matricial form, that is:

$$A := \begin{pmatrix} 1/2 & 0 \\ 0 & 1/\beta \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1/2 \\ 1/\beta^2 \end{pmatrix}.$$

Let $\mathbf{A} : \Omega := \{\mathbf{0}, \mathbf{1}\}^{\mathbf{N}} \to \mathbf{R}^2$ be the map such that $\mathbf{A}(\xi)$ is the limit of $A_{\xi_0 \cdots \xi_{n-1}}(0,0)$, when n goes to infinity that is, $\mathbf{A}(\xi) = (\mathbf{X}(\xi), \mathbf{Y}(\xi))$, where

$$\mathbf{X}(\xi) = \sum_{k=0}^{\infty} \xi_k / 2^{k+1} \quad \text{and} \quad \mathbf{Y}(\xi) = (\beta - 1) \sum_{k=0}^{\infty} \xi_k / \beta^{k+1}.$$

It is easily seen that $\mathbf{Y}(\xi) = \mathcal{W}_\beta(\mathbf{X}(\xi))$, where $\mathcal{W}_\beta$ is the Weierstrass-like function associated to the $\mathbf{Z}$-periodic function $\varphi : x \mapsto ([2x] - 2[x])/(1 - 1/\beta)$ (the coefficient $1 - 1/\beta$ is introduced for notational convenience). In particular, this means that $\mathbf{A}(\Omega)$ coincides with the graph Γ_β of $\mathcal{W}_\beta$ or equivalently, that Γ_β is the self-affine set such that $\Gamma_\beta = A_{\mathbf{0}}(\Gamma_\beta) \bigcup A_{\mathbf{1}}(\Gamma_\beta)$. There exists a natural expanding transformation $T : \Gamma_\beta \to \Gamma_\beta$, whose inverse branches are the affine contractions $A_{\mathbf{0}}$ and $A_{\mathbf{1}}$ meaning that $T(x, \mathcal{W}_\beta(x)) = A_i^{-1}(x, \mathcal{W}_\beta(x))$, with $i = \mathbf{0}$ if $0 \leq x \leq 1/2$ and $i = \mathbf{1}$ otherwise. In order to study the fractal geometry of Γ_β, it is worth to consider the *good* positive measure supported by Γ_β: here we define the distribution μ_β of the random variable $\mathbf{A} : \Omega \to \mathbf{R}^2$ when Ω is weighted by the uniform Bernoulli measure $\mathbb{P}$. The probability measure μ_β is T-ergodic (with metric entropy $\mathrm{h}_{\mu_\beta}(T) = \log 2$) and according to Hutchinson Theorem [15], [6, Thm. 2.8], it is the unique probability measure satisfying the self-affine equation

$$\mu_\beta = \frac{1}{2} \mu_\beta \circ A_{\mathbf{0}}^{-1} + \frac{1}{2} \mu_\beta \circ A_{\mathbf{1}}^{-1}. \tag{2}$$

Now, consider the orthogonal projections $\pi_x : \mathbf{R}^2 \to \mathbf{R} \times \{0\}$ and $\pi_y : \mathbf{R}^2 \to \{0\} \times \mathbf{R}$. (In what follows $\mathbf{R} \times \{0\}$ and $\{0\} \times \mathbf{R}$ are both identified to $\mathbf{R}$.) On the one hand, $\mu_\beta \circ \pi_x^{-1}$ is the distribution of the random variable $\mathbf{X}$ on $(\Omega, \mathbb{P})$ that is, the restriction