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by

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Abstract. — These notes were used as part of the Spring School "École Plurithématique de Théorie Ergodique II" Luminy, April 2006. They focus on the properties of unimodal maps, their description in terms of kneading maps, and the behavior of the unimodal map restricted to the omega-limit set $\omega(c)$ of its critical point if this is a Cantor set. Major references are [2, 7, 8, 20].

Résumé **(Combinatoire pour les applications unimodales (du type de Fibonacci))**
Ces notes ont été utilisées dans les cours de l'École Plurithématique de Théorie Ergodique II, Luminy, Avril 2006. Elles traitent les propriétés des applications unimodales, leur description en termes d'applications de pétrissage, et le comportement d'une application unimodale restreinte à l'ensemble ω-limite $\omega(c)$ de son point critique dans le cas où $\omega(c)$ est un ensemble de Cantor. Les réferences principales sont [2, 7, 8, 20].

1. Combinatorics of Unimodal Maps

A unimodal map $f : I \to I$ on the interval is a continuous map having a unique point c, the *critical point*, such that f is increasing to the left and decreasing to the right of c. Let $c_n = f^n(c)$ be the n-th image of the critical point. It is convenient to scale f such that the interval coincides with the *core*: $I = [c_2, c_1]$, and unless $c_2 < c < c_1$ and $c_2 \leq c_3$, the dynamics of f are not very interesting.

The results that we state here hold for the family of unimodal maps

$$f_a(x) = 1 - a|x|^\ell, \quad a \in [0, 2].$$

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Here $c = 0$ is the critical point and ℓ is the order of the critical point. If $\ell = 1$, then f_a is the tent family; if $\ell = 2$ then f_a is the quadratic family. The core of f_a is the interval $I = [c_2, c_1] = [1 - a, 1]$. If $a \in [1, 2]$, then f_a is onto on this interval; if $a < 1$, then every point in $[-1, 1]$ is attracted to a fixed point.

1.1. Symbolic dynamics. — The system (I, f) can be described symbolically by a subshift of $\{0, \star, 1\}^{\mathbb{N}}$ where each $x \in I$ is assigned an *itinerary* $i(x) = i_0(x)i_1(x)i_2(x)\dots$ where

$$i_k(x) = \begin{cases} 0 & \text{if } f^k(x) \in [c_2, c), \\ \star & \text{if } f^k(x) = c, \\ 1 & \text{if } f^k(x) \in (c, c_1]. \end{cases}$$

If Σ is the collection of all itineraries, and σ is the left-shift, then the below diagram commutes.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "i"'] & I \arrow[d, "i"] \\ \Sigma \arrow[r, "\sigma"'] & \Sigma \end{tikzcd}$$

Take $x \notin \operatorname{orb}^-(c) := \cup_{j \geq 0} f^{-j}(c)$, $i(x) \in \{0, 1\}^{\mathbb{N}}$. For each k, the set

$$J_k(x) := \{y \in I : i_0(y) \dots i_{k-1}(y) = i_0(x) \dots i_{k-1}(x)\}$$

is an open interval; it is a maximal open neighborhood on which f^k is monotone. It can happen that there are several points with the same itinerary. In this case, $H = \cap_k J_k(x)$ is a non-degenerate interval; it is called a *homterval*, because $f^k : H \to f^k(H)$ is a homeomorphism for every k. If f is a *non-flat* (*i.e.*, the critical order is finite) C^2-map, then any homterval is attracted to a periodic orbit or interval, [20]. This has the following convenient consequence:

Lemma 1. — *If f has no wandering intervals or periodic attractor, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if J is an interval of length $|J| > \delta$, then $|f^n(J)| > \varepsilon$ for all $n \geq 0$.*

Proof. — See [20, Chapter IV], where this property is called the *Contraction Principle*, although *Non-contraction Principle* seems a better word. $\square$

The *kneading invariant* is defined as the itinerary of the critical point, leaving out the initial $\star$:

$$\nu = \nu_f = \nu_1 \nu_2 \nu_3 \dots$$

Two unimodal maps are *combinatorially equivalent* if they have the same kneading invariant. If f and g are topologically conjugate, then they are combinatorially equivalent, but the converse is not true. The kneading invariant fails to notice:

– *Inessential periodic attractors*, *i.e.*, periodic attractors that don't attract the critical point. Recall that if f has negative Schwarzian derivative, or more precisely $Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\frac{f'''(x)}{f'(x)} \leq 0$ for every non-critical point x, then every periodic attractor attracts a critical point or boundary point, see [21]. In our case, we restricted the map to the core, so the boundary points belong to the critical orbit. Hence in this setting, every periodic attractor is essential.

– Wandering intervals, which however don't exist if f is non-flat and C^2, see [20, Chapter IV].

– The precise period of a periodic attractor generated in a period doubling bifurcation. For example, if $a_1 < a_2$ are parameters just before and after the first period doubling bifurcation creating a periodic attractor of period 2. Then in both cases $\nu_f = 1111\dots$, regardless whether $\omega(c)$ consist of a single or two points in $(c, c_1]$. The kneading invariant indicates this difference in period only when one of these period 2 points passes through c, as parameter a increases.

Two itineraries i and $i^\flat$ can be compared in *parity lexicographical order* $\prec_p$. First set $0 < \star < 1$. If $k = \min\{j \geq 0 : i_j \neq i^\flat_j\}$ then

$$i \prec_p i^\flat \text{ if } \begin{cases} i_k < i^\flat_k & \text{and } \#\{j < k : i_j = 1\} \text{ is even,} \\ i_k > i^\flat_k & \text{and } \#\{j < k : i_j = 1\} \text{ is odd.} \end{cases}$$

***Lemma 2* (See [8]).** — *The map $i : I \to \Sigma$, $x \mapsto i(x)$ is order preserving.*

Corollary 1. — *Given a unimodal map f with kneading invariant ν,*

$$\sigma(\nu) \preceq_p i(x) \preceq_p \nu \text{ for all } x \in I. \tag{1}$$

and

$$\sigma(\nu) \preceq_p \sigma^n(\nu) \preceq_p \nu \text{ for all } n \geq 0. \tag{2}$$

Conversely, we have:

Lemma 3

– *Fix $f : I \to I$ with kneading invariant ν. If $e \in \{0,1\}^{\mathbb{N}}$ is a sequence such that (1) holds, then there is $x \in I$ such that $i(x) = e$.*

– *If $\nu \in \{0,1\}^{\mathbb{N}}$ is a sequence such that (2) holds, then there is a unimodal map f such that $\nu = \nu_f$.*

For this reason, equation (2) is called the *admissibility condition* for kneading invariants. A map f is *renormalizable* if there is an interval $J \ni c$ and period p such that $f^p(J) \subset J$ and $f^i(J)$ and $f^j(J)$ have disjoint interiors for $0 \le i < j < p$. In this case, the map $f^p : J \to J$ is a new unimodal map, which can be renormalizable itself. Continuing inductively, we can arrive at *infinitely renormalizable* maps which have an infinite sequence of nested periodic interval $J_n \ni c$ of periods $p_n \to \infty$. The best known example is the *Feigenbaum-Coullet-Tresser map* (usually called *Feigenbaum map*) which has a periodic interval J_n of period 2^n for each $n \in \mathbb{N}$.

Renormalizability can be seen from the structure of the kneading invariant by the fact the ν has the structure of a *star-product.*

Proposition 1. — *Let f have a p-periodic interval J such that the itinerary of c starts with $\star i_1 \dots i_{p-1}$. Let $f^p : J \to J$ be a unimodal map with kneading invariant $\tilde\nu$. Then the kneading invariant of f itself is*

$$\nu = \begin{cases} i_1 \dots i_{p-1}\tilde\nu_1 i_1 \dots i_{p-1}\tilde\nu_2 i_1 \dots i_{p-1}\tilde\nu_3 \dots & \text{if } \#\{j < k : i_j = 1\} \text{ is even,} \\ i_1 \dots i_{p-1}\tilde\nu'_1 i_1 \dots i_{p-1}\tilde\nu'_2 i_1 \dots i_{p-1}\tilde\nu'_3 \dots & \text{if } \#\{j < k : i_j = 1\} \text{ is odd.} \end{cases} \tag{3}$$

Here $\tilde\nu'_k = 1, \star, 0$ if $\tilde\nu_k = 0, \star, 1$ respectively.

The sequence ν defined by (3) is known as the *star-product* of $\star i_1 \dots i_{p-1}$ and $\tilde\nu$, and written as $\nu = (\star i_1 \dots i_{p-1}) * \tilde\nu$, see [8].

1.2. Cutting times. — If J is a maximal (closed) interval on which f^n is monotone, then $f^n : J \to f^n(J)$ is called a *branch*. If $c \in \partial J$, $f^n : J \to f^n(J)$ is a *central branch*. Obviously f^n has two central branches, and they have the same image if n is sufficiently large. Denote this image (or the largest of the two) by D_n.

If $D_n \ni c$, then n is called a *cutting time*. Denote the cutting times by $\{S_i\}_{i\ge 0}$, $S_0 < S_1 < S_2 < \dots$ For interesting unimodal maps (such as tent maps with slope > 1 or f_a with $a \in (1,2]$) $S_0 = 1$ and $S_1 = 2$.

Lemma 4. — *Let $\beta(n) = n - \max\{S_k : S_k < n\}$. Then*

$$D_n = [c_n, c_{\beta(n)}] \text{ or } [c_{\beta(n)}, c_n] \quad \text{for all } n \ge 2, \tag{4}$$

and $D_n \subset D_{\beta(n)}$.

Proof. — For simplicity write $[x, y]$ for the interval with endpoints x and y, even if $y < x$. We prove (4) by induction. Since $D_2 = [c_2, c_1]$, it holds for $n = 2$. Next assume that (4) holds for n. If $D_n \not\ni c$ (so n is not a cutting time), then $D_{n+1} = f(D_n) = [c_{n+1}, c_{1+\beta(n)}]$. But $\beta(n+1) = n+1-\max\{S_k : S_k < n+1\} = n+1-\max\{S_k : S_k < n\} = 1+\beta(n)$. So the above interval is $[c_{n+1}, c_{\beta(n+1)}]$. If on the other hand $D_n \ni c$, then $D_{n+1} = [c_{n+1}, c_1]$, but $\beta(n+1) = 1$, so (4) holds for $n+1$. This proves the first statement.