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by

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Abstract. — This paper contains short lecture notes on joinings of rank-one transformations. Using the approximation of self-joinings by off-diagonal measures we prove King's theorems on two-fold minimal self-joinings (§1) and the weak closure theorem for rank-one **Z**-actions (and flows) (§2). Higher order self-joinings are also approximated by off-diagonal measures, and this gives the connection between multiple mixing and minimal self-joinings (§3). Blum-Hanson's ergodic theorem for mixing transformations and Kalikow's lemma on microreturns of the blocks (§4) are used in the joining proof of Kalikow's theorem on 3-fold mixing for rank-one transformations (§5).

***Résumé* (Auto-couplages d'actions de rang 1 et applications).** — Cet article est constitué de courtes notes sur l'auto-couplage des transformations de rang un. En utilisant l'approximation des auto-couplages par les mesures hors diagonale nous démontrons les théorèmes de King sur les auto-couplages minimaux à deux feuilles (§1) et le théorème de fermeture faible pour les **Z**-actions de rang un (et les flots) (§2). Des auto-couplages d'ordre plus Ãl'levÃl's sont aussi approchÃl's par des mesures hors diagonale ce qui donne un lien entre le mixage multiple et les autocouplages mininaux (§3). Le thÃl'orÃÍme ergodique de Blum-Hanson pour les transformations mÃl'langeantes et le lemme de Kalikow sur les micro-retours des blocks (§4) sont utilisÃl's dans la dÃl'monstration du thÃl'orÃÍme de Kalikow sur le 3-mÃl'lange pour les transformations de rang un (§5).

The first examples of rank-one transformations appeared in the well-known works by R. Chacon (geometric constructions), A. Katok, V. Oseledets, A. Stepin (theory of periodic approximation), D. Ornstein (stochastic constructions). Rank one mixing transformations have no factors and commute only with their powers (D. Ornstein). D. Rudolph gave an example of rank-one mixing transformation with extreme properties (minimal self-joinings) for his machinery of counterexamples. Rank one mixing transformations possess multiple mixing [7, 13] and they have to have the property

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of minimal self-joinings (J. King, see [9]). Let us remark that all known examples of $\mathbf{Z}$-actions with minimal self-joinings (MSJ) are of rank one. In [12] A. Prikhod'ko announced the existence of infinite rank transformations with MSJ. Note also that some (infinite rank) horocyclic flows have MSJ (M. Ratner, see [15]). Some spectral properties of rank-one mixing transformations are studied in [1, 4] (stochastic constructions) and [10, 14] (staircase constructions). We refer a reader to the bibliography presented in the articles [2, 9, 13] and the surveys [5, 6, 15].

Examples of rank-one actions of unusual commutative (and noncommutative) groups were built first by A. del Junco, then by A.I. Danilenko and C. Silva (see [2] and references therein). Several theorems on rank-one transformations can be lost below the horizon of $\mathbf{Z}, \mathbf{R}$-actions. There is a loss even in the case of $\mathbf{Z}^2$-actions [3]. In addition, T. Downarowicz recently constructed a partially mixing rank-one $\mathbf{Z}^2$-action with non-trivial factors. However, all partially mixing rank-one $\mathbf{Z}$-actions have no factors [8], and moreover, have minimal self-joinings [9]. It seems that the zoo of examples in [2] anticipate some "modern" theory which could be in contrast with "classical" one. Let us speak about the latter.

1. Rank one transformation. The approximation of self-joinings by off-diagonal measures. Mixing and two-fold minimal self-joinings

We consider probability Lebesgue space (X, μ). An automorphism (a measure-preserving invertible transformation) $T : X \to X$ is said to be of *rank one*, if there is a sequence ξ_j of measurable partitions of X in the form

$$\xi_j = \{E_j,\ TE_j,\ T^2E_j,\ \ldots, T^{h_j}E_j, \tilde{E}_j\}.$$

such that ξ_j converges to the partition onto points. The collection

$$E_j, TE_j,\ T^2E_j,\ \ldots, T^{h_j}E_j$$

is called Rokhlin's tower ($\tilde{E}_j$ is the set $X \setminus \bigsqcup_{i=0}^{h_j} T^iE_j$).

A *self-joining* (of order 2) is defined to be a $T \times T$-invariant measure ν on $X \times X$ with the marginals equal to μ:

$$\nu(A \times X) = \nu(X \times A) = \mu(A).$$

The joining ν is called ergodic if the dynamical system $(T \times T, X \times X, \nu)$ is ergodic.

The measures $\Delta^i = (Id \times T^i)\Delta$ (so called off-diagonals measures) are defined by the formula

$$\Delta^i(A \times B) = \mu(A \cap T^iB).$$

If T is ergodic, then Δ^i are ergodic self-joinings. We say that T has *minimal self-joinings* of order 2 (and we write $T \in MSJ(2)$) if T has no ergodic joinings except of $\mu \times \mu$ and Δ^i.

We say that T is *mixing*, if

$$\Delta^i \longrightarrow \mu \times \mu, \quad i \longrightarrow \infty,$$

i.e., for all measurable A, B

$$\Delta^i(A \times B) = \mu(A \cap T^i B) \longrightarrow \mu \times \mu(A \times B) = \mu(A)\mu(B).$$

Theorem 1.1. — *The mixing rank-one transformation T has minimal self-joinings of order two.*

Corollary 1.2. — *A mixing rank-one transformation commutes only with its powers and has no factors (*i.e.*, no non-trivial T-invariant σ-algebras).*

Proof of Corollary. — Suppose the automorphism S commutes with T. The joining $\Delta_S = (Id \times S)\Delta$ is ergodic: the system $(T \times T, X \times X, \Delta_S)$ is isomorphic to the ergodic system (T, X, μ). Then for some i we get

$$(Id \times S)\Delta = (Id \times T^i)\Delta,$$

this implies $S = T^i$. □

Let P be the orthoprojection operator onto the space $L_2(X, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a factor algebra (a T-invariant σ-subalgebra). Let us define a measure ν on $X \times X$ by setting

$$\nu(A \times B) = \int_X P\chi_A \ \chi_B \, d\mu.$$

Since P commutes with T we obtain $\nu(A \times B) = \nu(TA \times TB)$, so ν is a self-joining. From $T \in MSJ(2)$ we see that

$$P = c\Theta + \sum c_k T^k,$$

where Θ, the orthoprojection onto the space of the constants, corresponds to the measure $\mu \times \mu$; the operators T^k correspond to the off-diagonal measures $(Id \times T^k)\Delta$. From $P^2 = P$ we see that $P = \Theta$ or $P = Id$. Thus the factor algebra $\mathcal{A}$ must be trivial.

Theorem 1.3. — *Let T be of rank-one and ν an ergodic self-joining. Then there is a sequence k_j such that $(Id \times T^{k_i})\Delta \to \frac{1}{2}\nu + \frac{1}{2}\nu'$ for some self-joining ν': for all measurable A, B*

$$\mu(A \cap T^{k_i} B) \longrightarrow \frac{1}{2}\nu(A \times B) + \frac{1}{2}\nu'(A \times B).$$

Corollary 1.4. — *Theorem 1.1.*

Proof of Corollary. — Let ν be an ergodic self-joining. T is mixing, hence, $(Id \times T^{k_i})\Delta \to \mu \times \mu$ as $k_j \to \infty$. From Theorem 1.3 we have: either $\nu = \mu \times \mu$, or $\nu = (Id \times T^k)\Delta$. Thus $T \in MSJ(2)$. □

Proof of Theorem 1.3. — Our strategy is following: first we prove that joining can be approximated by sums of parts of the off-diagonal measures, then applying the Choice Lemma we find a sequence of parts tending to ν.

Given $\delta > 0$, $0 \le k \le \delta h_j$, we define the sets C_j^k (called columns):

$$C_j^k = \bigsqcup_{i=0}^{h_j - k} T^i T^k E_j \times T^i E_j.$$

For negative k $(-\delta h_j \le k \le 0)$, we put

$$C_j^k = \bigsqcup_{i=0}^{h_j + k} T^i E_j \times T^i T^{-k} E_j.$$

Let us consider the set

$$D_j^\delta = \bigsqcup_{k=-[\delta h_j]}^{[\delta h_j]} C_j^k. \tag{1}$$

For $\delta > \frac{1}{2}$ we have

$$\nu(D_j^\delta) > 1 - 2(1-\delta) = 2\delta - 1 > 0.$$

The sets D_j^δ are asymptotically $T \times T$-invariant, this implies

$$\nu(\ \mid D_j^\delta) \longrightarrow \nu,$$

since the limit measure is invariant and absolutely continuous with respect to ergodic ν.

Now we have

$$\sum_k \nu(C_j^k | D_j^\delta) \nu(\ | C_j^k) \longrightarrow \nu.$$

If A_j, B_j are ξ_j-measurable, then

$$\nu(A_j \times B_j | C_j^k) = \Delta^k (A_j \times B_j | C_j^k).$$

The density of the projections of the measures $\Delta^k(\ | C_j^k)$ and $\nu(\ | C_j^k)$ are bounded by $(1-\delta)^{-1}$. For arbitrary measurable sets A, B we can find ξ_j-measurable sets A_j, B_j such that

$$\varepsilon_j = \mu(A \Delta A_j) + \mu(B \Delta B_j) \longrightarrow 0.$$