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UNCOUNTABLE COLLECTION OF MIXING RANK-ONE ACTIONS FOR LOCALLY NORMAL GROUPS

by

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Abstract. — Let G be a discrete countable locally normal group. We construct an uncountable family of pairwise disjoint mixing (of any order) rank-one strictly ergodic free actions of G on a Cantor set. All of them possess the property of minimal self-joinings (of any order). Moreover, an example of partially rigid weakly mixing rank-one strictly ergodic free G-action is given.

Résumé **(Famille non dénombrable d'actions mélangeantes de rang un pour groupes localement normaux)**

Soit G un groupe discret, dénombrable, et localement normal. Nous construisons une famille non dénombrable d'actions de G sur un ensemble de Cantor, qui sont mélangeantes (à tous les ordres), mutuellement disjointes, libres de rang un et strictement ergodiques. Elles ont toutes des auto-couplages minimaux à tous les ordres. En outre, nous présentons une G-action libre, strictement ergodique, de rang un, faiblement mélangeante et partiellement rigide.

1. Introduction and definitions

Continuing our investigation from [5] we consider here the following problem:

Problem. — *Which countable discrete amenable groups G have mixing (funny) rank one free actions?*

Recall that a measure preserving action $T = (T_g)_{g\in G}$ of G on a standard probability space $(X, \mathfrak{B}, \mu)$ is called

— *mixing* if $\lim_{g\to\infty} \mu(A \cap T_g B) = \mu(A)\mu(B)$ for all $A, B \in \mathfrak{B}$,

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— *mixing of order l* if for any $\varepsilon > 0$ and $A_0, \dots, A_l \in \mathfrak{B}$, there exists a finite subset $K \subset G$ such that

$$|\mu(T_{g_0}A_0 \cap \dots \cap T_{g_l}A_l) - \mu(A_0)\cdots\mu(A_l)| < \varepsilon$$

for each collection $g_0, \dots, g_l \in G$ with $g_i g_j^{-1} \notin K$ if $i \neq j$,

— *weakly mixing* if the diagonal action $T \times T := (T_g \times T_g)_{g\in G}$ of G on the product space $(X \times X, \mathfrak{B} \otimes \mathfrak{B}, \mu \times \mu)$ is ergodic,

— *rigid* if there exists a sequence $g_n \to \infty$ in G such that

$$\lim_{n\to\infty} \mu(A \cap T_{g_n}B) = \mu(A \cap B) \quad \text{for all } A, B \in \mathfrak{B},$$

— *at least δ-partially rigid* (for some $0 < \delta \leq 1$) if there exists a sequence $g_n \to \infty$ in G such that

$$\lim_{n\to\infty} \mu(A \cap T_{g_n}B) \geq \delta\mu(A \cap B) \quad \text{for all } A, B \in \mathfrak{B}.$$

We say that T has *funny rank one* if there exist a sequence of measurable subsets $(A_n)_{n=1}^\infty$ in X and a sequence of finite subsets $(F_n)_{n=1}^\infty$ in G such that the subsets $T_g F_n$, $g \in F_n$, are pairwise disjoint for any n and

$$\lim_{n\to\infty} \min_{H\subset F_n} \mu\Big(B \triangle \bigsqcup_{g\in H} T_g A_n\Big) = 0 \quad \text{for every } B \in \mathfrak{B}.$$

If, moreover, $(F_n)_{n=1}^\infty$ is a subsequence of some 'natural' Følner sequence in G, we say that T has *rank one*. For instance, if $G = \mathbb{Z}^d$, this 'natural sequence' is just the sequence of cubes; if $G = \bigcup_{i=1}^\infty G_i$, every G_i is a finite group and $G_i \subset G_{i+1}$ then the sequence $(G_i)_{i=1}^\infty$ is 'natural', etc.

Up to now various examples of mixing rank-one actions were constructed for

— $G = \mathbb{Z}$ in [15], [17], [1], [3], etc.,
— $G = \mathbb{Z}^2$ in [2],
— $G = \mathbb{R}$ in [16], [10],
— $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ in [9],
— $G = \bigoplus_{i=1}^\infty G_i$, where G_i is a finite group in [5].

In the present paper we extend the latter result to the more general class of discrete countable *locally normal* groups G. This means that $G = \bigcup_{i=1}^\infty G_i$, where $G_1 \subset G_2 \subset \cdots$ is a nested sequence of normal finite subgroups of G. We call such a sequence a *filtration* of G. Of course, the class of locally normal groups includes the countable direct sums of finite groups and all Abelian torsion groups. Now we state the main result of the present paper (cf. with [5, Theorem 0.1]).

Theorem 1.1. — (i) *There exists an uncountable family of pairwise disjoint (and hence pairwise non-isomorphic) mixing rank-one strictly ergodic actions of G on a Cantor set. Moreover, these actions are mixing of any order.*

(ii) *There exists a weakly mixing and at least 0.5-partially rigid (and hence non-mixing) rank-one strictly ergodic action of G on a Cantor set.*

Concerning (i), it is worth to note that any mixing rank-one $\mathbb{Z}$-action is mixing of any order by [14] and [18] (see also an extension of that to actions of some Abelian groups with elements of infinite order in [13]). We do not know whether this fact holds for all mixing rank-one action of locally normal groups.

To prove the theorem, we combine the original Ornstein's idea of 'random spacer' (in the cutting-and-stacking construction process) [15] and the more recent (C,F)-construction developed in [11], [4]–[5], [8], [9]. As in [5], on the n-th step we cut the n-'column' into 'subcolumns' and then rotate each 'subcolumn' in a 'random way'. No spacers are added at all. In the limit we obtain a topological G-action on a compact Cantor space.

Our next concern is to describe all ergodic self-joinings of the G-actions constructed in Theorem 1.1. Recall a couple of definitions.

Given two ergodic G-actions T and T' on $(X,\mathfrak{B},\mu)$ and $(X',\mathfrak{B}',\mu')$ respectively, we denote by $J(T,T')$ the set of *joinings* of T and T', i.e. the set of $(T_g\times T'_g)_{g\in G}$-invariant measures on $\mathfrak{B}\otimes\mathfrak{B}'$ whose marginals on $\mathfrak{B}$ and $\mathfrak{B}'$ are μ and μ' respectively. The corresponding dynamical system $(X\times X',\mathfrak{B}\otimes\mathfrak{B}',\mu\times\mu')$ is also called a joining of T and T'. By $J^e(T,T')\subset J(T,T)$ we denote the subset of ergodic joinings of T and T' (it is never empty). In a similar way one can define the joinings $J(T_1,\dots,T_l)$ for any finite family $T_1,\dots,T_l$ of G-actions. If $J(T_1,\dots,T_l)=\{\mu_1\times\cdots\times\mu_l\}$ then the family $T_1,\dots,T_l$ is called *disjoint*. If $T_1=\cdots=T_l$ we speak about *l-fold self-joinings* of T_1 and use notation $J_l(T)$ for $J(\underbrace{T,\dots,T}_{l\text{ times}})$.

For $g\in G$, we denote by $g^\bullet$ the conjugacy class of g. Notice that $g^\bullet$ is always finite. We define a measure $\mu_{g^\bullet}$ on $(X\times X,\mathfrak{B}\otimes\mathfrak{B})$ by setting

$$\mu_{g^\bullet}(A\times B):=\frac{1}{\#g^\bullet}\sum_{h\in g^\bullet}\mu(A\cap T_hB).$$

It is easy to verify that $\mu_{g^\bullet}$ is a self-joining of T. Moreover, the map $(x,T_h^{-1}x)\mapsto(x,h)$ is an isomorphism of $(X\times X,\mu_{g^\bullet},T\times T)$ onto $(X\times g^\bullet,\mu\times\nu,\widetilde{T})$, where ν is the equidistribution on $g^\bullet$ and the G-action $\widetilde{T}=(\widetilde{T}_t)_{t\in G}$ is given by

$$\widetilde{T}_t(x,h)=(T_tx,tht^{-1}),\ x\in X,\ h\in g^\bullet.$$

It follows that $\widetilde{T}$ (and hence the self-joining $\mu_{g^\bullet}$ of T) is ergodic if and only if the action $(T_t)_{t\in C(g)}$ is ergodic, where $C(g)=\{t\in G\mid tg=gt\}$ stands for the centralizer of g in G. Notice also that $C(g)$ is a co-finite subgroup of G. Hence $\{\mu_{g^\bullet}\mid g\in G\}\subset J_2^e(T)$ whenever T is weakly mixing.

Definition 1.2 **([5])**. — If $J_2^e(T) \subset \{\mu_{g^\bullet} \mid g \in G\} \cup \{\mu \times \mu\}$ then we say that T has *2-fold minimal self-joinings* (MSJ_2). (In [6] this property is called near MSJ_2.)

This definition extends naturally to higher order self-joinings as follows. Given $l \geq 1$ and $g \in G^{l+1}$, we denote by $g^{\bullet l}$ the orbit of g under the G-action on G^{l+1} by conjugations:

$$h \cdot (g_0, \dots, g_l) := (hg_0h^{-1}, \dots, hg_lh^{-1}).$$

Let P be a partition of $\{0, \dots, l\}$. For an atom $p \in P$, we denote by i_p the minimal element in p. We say that an element $g = (g_0, \dots, g_l) \in G^{l+1}$ is *P-subordinated* if $g_{i_p} = 1_G$ for all $p \in P$. For any such g, we define a measure $\mu_{g^{\bullet l}}$ on $(X^{l+1}, \mathfrak{B}^{\otimes(l+1)})$ by setting

$$\mu_{g^{\bullet l}}(A_0 \times \cdots \times A_l) := \frac{1}{\# g^{\bullet l}} \sum_{(h_0,\dots,h_l) \in g^{\bullet l}} \prod_{p \in P} \mu\Big(\bigcap_{i \in p} T_{h_i} A_i\Big).$$

Then $\mu_{g^{\bullet l}}$ is an $(l+1)$-fold self-joining of T. It is ergodic whenever T is weakly mixing.

Definition 1.3 **([5])**. — We say that T has *$(l+1)$-fold minimal self-joinings* (MSJ_{l+1}) if

$$J_{l+1}^e(T) \subset \{\mu_{g^{\bullet l}} \mid g \text{ is } P\text{-subordinated for a partition } P \text{ of } \{0, \dots, l\}\}.$$

If T has MSJ_l for any $l > 1$, we say that T has MSJ.

In case G is Abelian, these definitions agree with the—common now—definitions of MSJ_{l+1} and MSJ by A. del Junco and D. Rudolph [12] who considered self-joinings $\mu_{g^{\bullet l}}$ only when g belongs to the center of G^{l+1}. However if G is non-Abelian then no ergodic action of G can have MSJ_2 in their sense.

As in [5] we deduce the following result from the proof of Theorem 1.1(i).

Proposition 1.4. — *The actions constructed in Theorem 1.1(i) all have MSJ.*

As was shown in [5] and [6] each action T with MSJ

— has trivial centralizer, i.e. $C(T) = \{T_g \mid g \in C(G)\}$,

— has trivial product centralizer,

— is effectively prime, i.e. for each non-trivial factor $\mathfrak{F}$ of T there exists a finite normal subgroup H in G such that

$$\mathfrak{F} = \{B \in \mathfrak{B} \mid T_g B = B \text{ for all } g \in H\}.$$

In particular, there exist no free factors of T.

We now briefly summarize the organization of the paper. In Section 2 we outline the (C, F)-construction of rank-one actions as it appeared in [4] (see also a recent survey [7]). In Section 3, we prove Theorem 1.1(i) and Proposition 1.4. Section 4 is devoted to the proof of Theorem 1.1(ii).