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A NOTE ON THE GREEDY β-TRANSFORMATION WITH ARBITRARY DIGITS

by

Karma Dajani & Charlene Kalle

Abstract. — We consider a generalization of the greedy and lazy β-expansions with digit set $A = \{a_0 < a_1 < \cdots < a_m\}$. We prove that the transformation generating such expansions admits a unique absolutely continuous invariant ergodic measure. Furthermore, the support of this measure is an interval.

***Résumé* (Note sur la β-transformation avec des chiffres arbitraires).** — Nous considérons une généralisation des « greedy » et « lazy » β-développements avec chiffres dans un alphabet $A = \{a_0 < a_1 < \cdots < a_m\}$. Nous montrons que la transformation qui donne ces développements possède une unique mesure qui soit invariante, ergodique et absolument continue par rapport à la mesure de Lebesgue. En outre, le support de cette mesure est un intervalle.

1. Introduction

Let $\beta > 1$ be a real number, and let T_β be the transformation of the unit interval [0,1) given by $T_\beta x = \beta x(\mathrm{mod} 1)$. This transformation gives rise to the β-expansion introduced by Rényi [19]: for any $0 \le x < 1$,

$$x = \sum_{k=1}^{\infty} d_k \beta^{-k} \, , \tag{1}$$

where $d_k = d_k(x, \beta) = \lfloor \beta T_\beta^{k-1} x \rfloor$, $k \ge 1$ (here $\lfloor \xi \rfloor$ denotes the greatest integer not exceeding ξ). Rényi showed that for each $\beta > 1$ the β-transformation T_β is ergodic, and that there exists a unique probability measure ν_β, equivalent to Lebesgue measure and invariant under T_β, such that for each Borel measurable set $B \in \mathcal{B}$ one has

$$\nu_\beta(B) = \int_B h_\beta(x)\, dx \, ,$$

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where $h_\beta(x)$ is a measurable function satisfying

$$1-\frac{1}{\beta} \leq h_\beta(x) \leq \frac{1}{1-\frac{1}{\beta}} .$$

Shortly afterwards, Parry [15] gave an explicit formula for the density of the invariant measure, namely,

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x<T^n(1)} \frac{1}{\beta^n} \; x \in [0,1)\,,$$

where $F(\beta) = \int_0^1 \left(\sum_{x<T^n(1)} \frac{1}{\beta^n}\right) dx$ is a normalizing constant.

The β-transformation given above can be defined geometrically in the following way. There exists a subdivision $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_m < \alpha_{m+1} = 1$ of $[0,1)$, such that T_β is linear with slope β on each subinterval $I_j = [\alpha_j, \alpha_{j+1})$. Further, $TI_j = [0,1)$, $T\alpha_j = 0$ for $j = 0,1,\cdots,m$, and $T1 = \lim_{x\to 1} Tx \leq 1$. This implies that on I_j, T is given by $Tx = \beta x - j$. Iterations of T give expansions of the form (1) with digit $d_k \in A = \{0,1,\cdots,m\}$ (notice that $m = \lfloor\beta\rfloor$).

Expansion (1) is also sometimes known in the literature by the *greedy expansion*. The reason is that, for each k the digit d_k in (1) is the largest element of $A = \{0,1,\cdots,\lfloor\beta\rfloor\}$ such that $\sum_{j=1}^k d_j\beta^{-j} \leq x$. So at each step, the greedy expansion chooses the largest possible digit. Taking this point of view, Pedicini [17] generalized the above notion to greedy expansions with digits in some set $A = \{a_0,\ldots,a_m\}$, with $a_0 < \cdots < a_m$. More precisely, he studied the combinatorial and arithmetic properties of expansions of the form $x = \sum_{k=1}^\infty d_k\beta^{-k}$ such that for each k, d_k is the largest element of $A = \{a_0,a_1,\ldots a_m\}$ with $\sum_{j=1}^k d_j\beta^{-j} \leq x$ (see [4] for more detail). He showed that every point in the interval $\left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a greedy expansion with digits in the set A, if and only if

$$\max_{0\leq j\leq m-1}(a_{j+1}-a_j) \leq \frac{a_m-a_0}{\beta-1}. \tag{2}$$

He called such expansions *greedy expansions with deleted digits.* This terminology has already been used by several authors (see [11], [18], [12], [17], [4], [21]), but in most cases the digit sets under consideration contain only non-negative integers. Since our digit sets will contain arbitrary real numbers, we choose to adopt *greedy expansions with arbitrary digits* instead or we will refer explicitly to the digit set that we use. We will now give a geometrical description of the underlying map generating these expansions, which shows why condition (2) imposed by Pedicini is a natural one. Furthermore, it allows us to put these transformations in the general framework of piecewise linear maps, for which a rich theory has been developed, and which we use to analyze and understand the ergodic properties of the β-transformations with arbitrary digits. The description is similar to the one given for the classical greedy expansion defined above. We consider transformations T whose domain is some interval $[0,\alpha]$ with the following properties

(i) there exists a subdivision $0=\alpha_0<\alpha_1<\cdots<\alpha_m<\alpha_{m+1}=\alpha$ with corresponding interval partition $J_j=[\alpha_j,\alpha_{j+1})$, $j=0,1,\dots m-1$, $J_m=[\alpha_m,\alpha]$ such that T on each J_j is linear with slope β,

(ii) $T\alpha_j=0$, $j=0,1,\dots,m$, and $T\alpha=\alpha$,

(iii) $TJ_j\subset[0,\alpha]$, $j=0,1,\dots m-1$.

Note that from (ii) and the linearity of T, we have that $TJ_m=[0,\alpha]$ so that T is surjective. Setting $a_j=\beta\alpha_j$ for $j=0,1,\dots m$, one sees from (i) and (ii) that on each J_j, T has the form $Tx=\beta x-a_j$, $j=0,1,\dots m$. From the second equation in (ii), one has that $\alpha=\frac{a_m}{\beta-1}$. From (iii) one gets $\max_{0\le j\le m-1}(a_{j+1}-a_j)\le\alpha=\frac{a_m}{\beta-1}$, which is condition (2) with $a_0=0$. See Figure 1 for the graph of T. Iterations of T generate greedy expansions with digits in A as described by Pedicini.

The above geometrical description can be slightly modified in order to capture the case $a_0\neq 0$. The interval $[0,\alpha]$ is replaced by $[\alpha_0,\alpha]$ ($0\neq\alpha_0<\alpha$), and in condition (ii) we replace the first equality by $T\alpha_j=\alpha_0$ for $i=0,1,\dots,m$. However, a simple translation by α_0 conjugates a transformation T in this class with domain $[\alpha_0,\alpha]$ and subdivision $\alpha_0<\alpha_1<\cdots<\alpha_m<\alpha_{m+1}=\alpha$ to a transformation S of the previous class on $[0,\alpha-\alpha_0]$ with subdivision $0=\gamma_0<\gamma_1<\cdots<\gamma_m<\gamma_{m+1}=\alpha-\alpha_0$, $\gamma_i=\alpha_i-\alpha_0$, $i=0,1,\dots,m+1$. Setting now $a_j=\beta\alpha_j-\alpha_0$, and using the conjugation with S or the defining properties of T, we see that

$$Tx=\begin{cases}\beta x-a_j, & \text{if } x\in\left[\dfrac{a_0}{\beta-1}+\dfrac{a_j-a_0}{\beta},\dfrac{a_0}{\beta-1}+\dfrac{a_{j+1}-a_0}{\beta}\right),\\ & \text{for } j=0,\dots,m-1,\\ \\ \beta x-a_m, & \text{if } x\in\left[\dfrac{a_0}{\beta-1}+\dfrac{a_m-a_0}{\beta},\dfrac{a_m}{\beta-1}\right].\end{cases}$$

We call $T=T_{\beta,A}$ the *greedy transformations with digits in the set* $A=\{a_0,\dots,a_m\}$ with $a_0<\cdots<a_m$. Clearly, A satisfies condition (2), and a set with this property is called an *allowable digit set for* β. From the above, and to keep the exposition simple, we will assume with no loss of generality that $a_0=0=\alpha_0$.

As explained above, the greedy expansion chooses at each stage the largest possible digit. One can look at the other extreme case, namely β-expansions $\sum_{k=1}^{\infty}d_k\beta^{-k}$ such that for each k, d_k is the smallest member of A satisfying

$$x\le\sum_{i=1}^{k}\frac{d_i}{\beta^i}+\sum_{i=k+1}^{\infty}\frac{a_m}{\beta^i}. \tag{3}$$

These expansions are known as *lazy expansions*, and were studied in the classical case, *i.e.*, $A = \{0, 1, \dots \lfloor\beta\rfloor\}$ by many authors (see for example [5], [6], [8] and [7]), and for the general case $A = \{a_0, \dots, a_m\}$ by Pedicini [17] (see also [4]). Pedicini showed that under condition (2), every $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a lazy expansion. In [4] the underlying transformation generating lazy expansions with digit set A was given, and was shown to be conjugate to the greedy expansion $T = T_{\beta,\tilde{A}}$ with $\tilde{A} = \{\tilde{a}_0, \dots, \tilde{a}_m\}$, where $\tilde{a}_i = a_0 + a_m - a_{m-i}$, $i \in \{0, \dots, m\}$. The isomorphism ϕ is given by

$$\begin{aligned} \phi : \quad \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right] &\longrightarrow \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right] \\ x &\longmapsto \frac{a_0+a_m}{\beta-1} - x. \end{aligned}$$

So we have $L \circ \phi = \phi \circ T$. The explicit definition of $L = L_{\beta,A}$ is given by

$$Lx = \begin{cases} \beta x - a_0, & \text{if } x \in \left[\dfrac{a_0}{\beta-1}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_0}{\beta}\right], \\[2ex] \beta x - a_j, & \text{if } x \in \left(\dfrac{a_m}{\beta-1} - \dfrac{a_m - a_{j-1}}{\beta}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_j}{\beta}\right], \\ & \text{for } j = 1, \dots, m. \end{cases}$$

As mentioned, throughout we will assume that $a_0 = 0$.

A number of articles have been published on invariant measures of piecewise monotonic transformations. Among others, the articles [1] by Buzzi and Sarig, [2] by Byers and Boyarsky, [3] by Byers, Góra and Boyarsky, [9] and [10] by Hofbauer, [13] by Lasota and Yorke, [14] by Li and Yorke, [20] by Schweiger and [22] by Wilkinson state a variety of results regarding invariant measures of this kind of transformations and their ergodicity.

In the first section of this article we will prove the existence of a unique absolutely continuous invariant ergodic measure μ for the greedy transformation with arbitrary digits using the results found by Li and Yorke in [14]. We will show that the support of this measure is the smallest interval of the form $[0, t)$ such $T([0, t)) \subset [0, t)$, and we identify t explicitly. This leads to the following theorem

Theorem 1.1. — *The restriction of T to the interval $[0, t)$ admits a unique invariant ergodic measure that is equivalent to Lebesgue measure on this interval.*

We give similar results for the lazy transformation with arbitrary digits. In the last section we consider in more detail two classes of greedy transformations with arbitrary digits, and we give an explicit formula for the density of their absolutely continuous invariant measures. For the first class an article by Wilkinson ([22]) has been an important source. For the second class we use an article by Byers and Boyarski ([2]), which is based on [16] by Parry.