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EXTREMALS FOR HARDY-SOBOLEV TYPE INEQUALITIES: THE INFLUENCE OF THE CURVATURE

by

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Abstract. — We consider the optimal Hardy-Sobolev inequality on a smooth bounded domain of the Euclidean space. Roughly speaking, this inequality lies between the Hardy inequality and the Sobolev inequality. We address the questions of the value of the optimal constant and the existence of non-trivial extremals attached to this inequality. When the singularity of the Hardy part is located on the boundary of the domain, the geometry of the domain plays a crucial role: in particular, the convexity and the mean curvature are involved in these questions. The main difficulty to encounter is the possible bubbling phenomenon. We describe precisely this bubbling through refined concentration estimates. An offshot of these techniques allows us to provide general compactness properties for nonlinear equations, still under curvature conditions for the boundary of the domain.

Résumé **(Extrémaux pour les inégalités Hardy-Sobolev : l'influence de la courbure)**

Nous considérons l'inégalité de Hardy-Sobolev optimale sur un domaine borné régulier de l'espace euclidien. Cette inégalité se situe entre l'inégalité de Hardy et celle de Sobolev. Nous abordons la question de l'optimalité des constantes attachées à cette inégalité ainsi que l'existence de solutions extrémales non triviales. Quand la singularité de la partie Hardy de l'inégalité est localisée sur le bord du domaine, la géométrie du domaine joue un rôle crucial: en particulier la convexité et la courbure moyenne sont impliquées dans ces questions. La principale difficulté à contourner est la possibilité d'existence de phénomène de concentration. Nous décrivons précisément ce type de phénomène par des estimées de concentration fines. Une ramification de ces techniques nous permet de fournir des propriétés générales de compacité pour des équations non linéaires, sous des conditions de courbure sur le bord.

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1. The Hardy-Sobolev inequality and two questions

We consider the Euclidean space $\mathbb{R}^n$, $n \geq 3$. The famous Sobolev theorem asserts that there exists a constant $C_1(n) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}} \leq C_1(n) \int_{\mathbb{R}^n} |\nabla u|^2\, dx \tag{1}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. Another very famous inequality is the Hardy inequality, which asserts that there exists $C_2(n) > 0$ such that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx \leq C_2(n) \int_{\mathbb{R}^n} |\nabla u|^2\, dx \tag{2}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. Interpolating these two inequalities, one gets the Hardy-Sobolev inequality: more precisely, let $s \in [0, 2]$, then there exists $C(s, n) > 0$ such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}} \leq C(s, n) \int_{\mathbb{R}^n} |\nabla u|^2\, dx \tag{3}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$, where

$$2^\star(s) := \frac{2(n-s)}{n-2}.$$

Indeed, with $s = 0$, we recover the Sobolev inequality (1), and with $s = 2$, we recover the Hardy inequality (2). The Hardy-Sobolev inequality is a particular case of the family of functional inequalities obtained by Caffarelli-Kohn-Nirenberg [8]. When $s \in (0, 2)$, it is remarkable that the Hardy-Sobolev inequality inherites the singularity at 0 from the Hardy inequality and the superquadratic exponent from the Sobolev inequality. For completeness and density reasons, given Ω an open subset of $\mathbb{R}^n$, it is more convenient to work in the Sobolev space

$$H_{1,0}^2(\Omega) := \text{ Completion of } C_c^\infty(\Omega) \text{ for } \|\cdot\|$$

where $\|u\| := \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}$. Therefore, inequality (3) is valid for $u \in H_{1,0}^2(\Omega)$.

Following the programme developed for other functional inequalities, we saturate (3): given Ω an open subset of $\mathbb{R}^n$, we define

$$\mu_s(\Omega) := \inf_{u \in H_{1,0}^2(\Omega) \setminus \{0\}} I_\Omega(u), \text{ where } I_\Omega(u) := \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)}.$$

It follows from the Hardy-Sobolev inequality that $\mu_s(\Omega) > 0$. We address the two following questions:

Question 1. — *What is the value of $\mu_s(\Omega)$?*

Question 2. — *Are there extremals for $\mu_s(\Omega)$?*

That is: is there some $u_\Omega \in H_{1,0}^2(\Omega) \setminus \{0\}$ such that $I_\Omega(u_\Omega) = \mu_s(\Omega)$?

The main difficulty here is due to the fact that $2^\star(s)$ is critical from the viewpoint of the Sobolev embeddings. More precisely, if Ω is bounded, then $H_{1,0}^2(\Omega)$ is embedded in the weighted space $L^p(\Omega, |x|^{-s})$ for $1 \leq p \leq 2^\star(s)$. And the embedding is compact iff $p < 2^\star(s)$ (in general, at least... see subsection 2.3 below). This lack of compactness defeats the classical minimization strategy to get extremals for $\mu_s(\Omega)$. In fact, when $s = 0$, that is in the case of Sobolev inequalities, the same kind of difficulty occurs, and there have been some methods developed to bypass them. Concerning the same questions in the Riemannian context, we refer to Hebey-Vaugon [23] and Druet [10], and also to Aubin-Li [4].

2. A few answers in some specific cases

In this section, we collect a few facts and answers to questions 1 and 2: these results are essentially extensions of the methods developed in the case $s = 0$.

2.1. The case $s = 0$. — In this context, the situation is well understood. In particular,

$$\mu_0(\mathbb{R}^n) = n(n-2)\left(\frac{\omega_{n-1}}{2}\cdot\frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}\right)^{\frac{2}{n}} = \frac{n(n-2)\omega_n^{2/n}}{4}$$

where ω_k is the volume of the standard $k-$sphere of $\mathbb{R}^{k+1}$ and Γ is the Gamma function. The extremals exist and are known: indeed, $u \in H_{1,0}^2(\mathbb{R}^n) \setminus \{0\}$ is an extremal for $\mu_0(\mathbb{R}^n)$ if and only if there exist $x_0 \in \mathbb{R}^n$, $\lambda \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$ such that

$$u(x) = \lambda\left(\frac{\alpha}{\alpha^2 + |x - x_0|^2}\right)^{\frac{n-2}{2}} \text{ for all } x \in \mathbb{R}^n. \quad (4)$$

These results are due to Rodemich [28], Aubin [3] and Talenti [30]. We also refer to Lieb [24] and Lions [25] for other nice points of view.

Concerning general open subsets of $\mathbb{R}^n$, one can show that

$$\mu_0(\Omega) = \mu_0(\mathbb{R}^n) = \frac{n(n-2)\omega_n^{2/n}}{4}$$

for all Ω open subset of $\mathbb{R}^n$. Moreover, if there is an extremal for $\mu_s(\Omega)$, then it is also an extremal for $\mu_0(\mathbb{R}^n)$ and it is of the form of (4). In particular, there is no extremal for $\mu_s(\Omega)$ if Ω is bounded (more general conditions involving the capacity are available).

From now on, we concentrate on the case $s \in (0, 2)$. Here, due to the singularity at 0, the situation will depend highly on the location of 0 with respect to Ω

2.2. The case $0 \in \Omega$, $s \in (0,2)$. — Here again, when $\Omega = \mathbb{R}^n$, the constant $\mu_s(\Omega)$ is explicit, and we know what the extremals are (see Ghoussoub-Yuan [19], Lieb [24], we refer also to Catrina-Wang [9]). More precisely,

$$\mu_s(\mathbb{R}^n) = (n-2)(n-s)\left(\frac{\omega_{n-1}}{2-s} \cdot \frac{\Gamma^2(\frac{n-s}{2-s})}{\Gamma(\frac{2n-2s}{2-s})}\right)^{\frac{2-s}{n-s}}$$

and given $\alpha > 0$, the functions

$$u_\alpha(x) := \left(\frac{\alpha^{\frac{2-s}{2}}}{\alpha^{2-s} + |x|^{2-s}}\right)^{\frac{n-2}{2-s}}$$

are extremals for $\mu_s(\mathbb{R}^n)$, and $u \in H^2_{1,0}(\mathbb{R}^n) \setminus \{0\}$ is an extremal for $\mu_s(\mathbb{R}^n)$ iff there exists $\lambda \in \mathbb{R} \setminus \{0\}$ and $\alpha > 0$ such that $u = \lambda \cdot u_\alpha$. When $s = 0$, we recover some of the extremals for the standard Sobolev inequality. Here, it is important to note the following asymptotics for u_α when $\alpha \to 0$:

$$\lim_{\alpha \to 0} u_\alpha(0) = +\infty \text{ and } \lim_{\alpha \to 0} u_\alpha(x) = 0 \text{ for all } x \neq 0.$$

In other words, the function u_α concentrates at 0 when $\alpha \to 0$.

When dealing with an open subset Ω of $\mathbb{R}^n$ such that $0 \in \Omega$, one can follow the approach developed for $s = 0$. Indeed, it follows from the definition of $\mu_s(\Omega)$ that

$$\mu_s(\Omega) \geq \mu_s(\mathbb{R}^n).$$

The reverse inequality is obtained via the estimate of I_Ω at a suitable test-function. Indeed, let $\eta \in C_c^\infty(\Omega)$ such that $\eta(x) \equiv 1$ in a neighborhood of 0. Then $\eta u_\alpha \in C_c^\infty(\Omega)$. Simple computations then yield

$$I_\Omega(\eta u_\alpha) = \mu_s(\mathbb{R}^n) + o(1)$$

where $\lim_{\alpha \to 0} o(1) = 0$. It then follows that $\mu_s(\Omega) \leq \mu_s(\mathbb{R}^n)$, and then

$$\mu_s(\Omega) = \mu_s(\mathbb{R}^n).$$

Indeed, this is exactly the standard proof in the case $s = 0$. Concerning the extremals, the same argument as for $s = 0$ proves that there is no extremal for $\mu_s(\Omega)$ if Ω is bounded. To conclude, one can say that the case $s \in (0,2)$ when $0 \in \Omega$ is quite similar to the case $s = 0$.

2.3. The case $0 \notin \overline{\Omega}$, $s \in (0,2)$. — This case is not the most interesting. Indeed, when $0 \notin \overline{\Omega}$ and Ω is bounded, then $L^{2^\star(s)}(\Omega, |x|^{-s}) = L^{2^\star(s)}(\Omega)$ and the embedding $H^2_{1,0}(\Omega) \hookrightarrow L^{2^\star(s)}(\Omega)$ is compact since $1 \leq 2^\star(s) < \frac{2n}{n-2}$. Therefore, the standard minimization methods work and there are extremals for $\mu_s(\Omega)$. However, finding the explicit value of $\mu_s(\Omega)$ is almost impossible in general.