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ANALYTIC ASPECTS OF PROBLEMS IN RIEMANNIAN GEOMETRY: ELLIPTIC PDES, SOLITONS AND COMPUTER IMAGING

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MEAN CURVATURE FLOW SOLITONS

by

Norbert Hungerbühler & Beatrice Roost

Abstract. — We consider the mean curvature flow $F_t : M \to N$ of hypersurfaces in a Riemannian manifold N. The stationary solutions of this flow are the minimal surfaces in N. Other interesting solutions are those, which move along the integral curves of a smooth vector field X of N. In this way conformal vector fields X give raise to self-similarly shrinking solutions of the mean curvature flow. If X is even parallel then the corresponding solutions of the mean curvature flow are called isometric solitons or just solitons. Soliton solutions have attracted increasing attention in the past years since they are interesting objects for a number of reasons: solitons appear as blow ups of singularities and exhibit interesting geometric and analytic properties. They serve as tailor-made comparison solutions and allow a certain insight into the behaviour of the mean curvature flow viewed as a dynamical system.

***Résumé* (Solitons issus du flot par la courbure moyenne).** — Nous considérons le flot de la courbure moyenne $F_t : M \to N$ d'hypersurfaces dans une variété riemannienne N. Les solutions stationnaires de ce flot sont les surfaces minimales dans N. D'autres solutions intéressantes sont celles qui se déplacent le long de courbes intégrales d'un champ de vecteur lisse X dans N. De cette manière les champs de vecteurs conformes X engendrent des solutions autosimilaires contractantes du flot de la courbure moyenne. Si X est parallèle alors les solutions correspondantes au flot de la courbure moyenne sont appelées solitons isométriques ou juste solitons. Il y a un intérêt croissant ces dernières années pour les solutions solitons car ce sont des objets intéressants pour diverses raisons: les solitons apparaissent comme des éclatements de singularités et font apparaître des propriétés géométriques et analytiques intéressantes. Elles servent comme des solutions de comparaison sur mesure et donnent une certaine idée du comportement du flot de la courbure moyenne vu comme un système dynamique.

1. Introduction

Physicists investigated in the fifties of the twentieth century the annealing process of aluminum. They observed, that in melted aluminum, at random points the material

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starts to crystallize spontaneously, as the temperature reaches a critical level. In these points, homogeneous crystals with face centered cubic lattice start to grow. These grains finally touch each other and fill the space (see Figure 1). However, this is not the end of the process: atoms sitting at the edge of a grain are integrated in their atomic

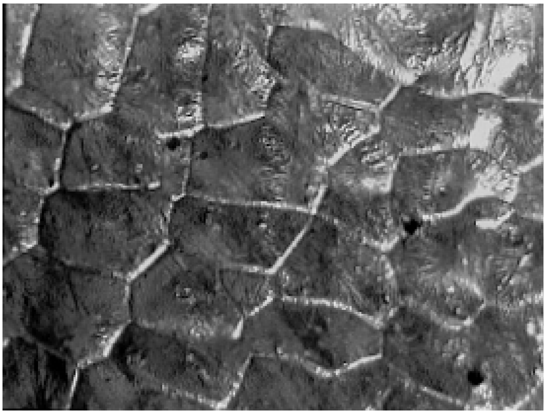

FIGURE 1. Grains in aluminum: a typical grain size is around 10 micrometer, the lattice parameter of aluminum amounts to 4.05 10^{-7}m.

crystal lattice only to one side and are therefore in a slightly elevated energy state. On account of this, such an atom can spontaneously jump to the neighboring lattice. This change is the more likely, the more convex the boundary at this point is: if, e.g., the atom is sitting at a cusp, it is surrounded almost entirely by a "foreign" crystal grid and will therefore easily change its affiliation. By the described mechanism, the grain boundaries keep moving even after the metal has solidified. It has been observed, that the velocity of a grain boundary is proportional to its mean curvature. This is plausible, if we assume that the elevated energy state of the boundary atoms amounts to a surface energy which is isotropic and proportional to the surface area. The system, trying to minimize its energy, will therefore reduce this surface, and the first variation of the area functional corresponds just to the mean curvature vector field. This means, the system reduces its energy by moving the grain boundaries with a velocity which is (proportional to) the mean curvature in each point. This is the *mean curvature flow*.

Mathematically, the mean curvature flow has first been investigated 1978 by Brakke (see [4]), later by Huisken (see [12]). Brakke used geometric measure theory, Huisken a more classical, differential geometric approach. In order to describe singularities of the flow, Osher-Sethian introduced a level-set formulation for the mean curvature flow (see [18]), which was investigated later by Evans-Spruck (see [6], [7], [8], [9]) and Chen-Giga-Goto (see [5]) in detail. Ilmanen revealed in [14] the relation between the level-set formulation and the geometric measure theory approach.

In this article, we use the following model of the mean curvature flow: let N be a n-dimensional Riemannian manifold with a metric $\bar{g}$ and M a differentiable, connected m-dimensional manifold with $m = n - 1$. Let $F_t : M \to N$, $t \in [0, T[, T > 0$, be a smooth family of immersions from M to N. Then we say:

Definition 1.1. — The family F_t is a solution of the mean curvature flow on $[0, T[$, $T > 0$, if

$$\begin{aligned} \frac{d}{dt}F_t &= -H\nu \quad \text{on } M\times]0,T[\\ F_0 &= f \qquad \text{on } M, \end{aligned} \tag{1}$$

where $f : M \to N$ describes a given initial hypersurface M_0. H denotes the mean curvature of $F_t(M)$ with respect to the unit normal vector field ν on $F_t(M)$. ◇

The minus sign in (1) causes the flow to decrease area (or arc length in the case of curves). We also remark that the product $H\nu$ is independent of the chosen orientation of ν (see (6)–(7) below). H can be interpreted as the trace of the second fundamental form of the immersion, and $H\nu$ as the first variation of the area functional. The term $H\nu$ can also be written as $\Delta_{F_t(M)}F_t$, which is the Laplace-Beltrami operator on M with respect to the pull back by F_t of the metric on N. In this form, the parabolic nature of the equation becomes apparent. However, the operator evolves in time together with the solution. Nonetheless, classical solutions of the mean curvature flow, inherit a parabolic comparison principle:

Comparison principle. — *If the initial surfaces $F_0(M)$ and $G_0(M')$ are disjoint, so are the solutions $F_t(M)$ and $G_t(M')$ as long as they exist classically.*

This comparison principle allows already to make some qualitative statements regarding the behaviour of solutions of the mean curvature flow. The following example in $N = \mathbb{R}^3$ is due to Angenent: the two spheres S in Figure 2 have the same radius

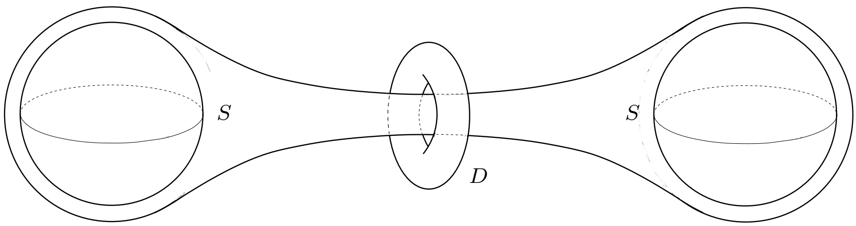

FIGURE 2. Four initial surfaces

R. Subject to the mean curvature flow, they shrink in time $T_S = \frac{R^2}{2\dim S}$ to a point. Between the two spheres, there is a special torus D which has the property, that it shrinks self similarly to a point. Such a torus has been found in 1992 by Angenent (Angenent's doughnut, see [3]). Choosing the torus small enough to be enclosed by a sphere of radius $r < R$, its vanishing time T_D is strictly less than T_S. Finally, we thread a dumbbell surface around the two spheres S through the torus D. The comparison principle guarantees that the configuration stays disjoint during its evolution under the mean curvature flow. Therefore, after a certain time, the solution looks as indicated in Figure 3. At the latest at time T_D the torus strangles the neck of the

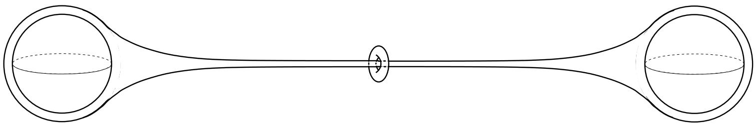

FIGURE 3. Solution at time $t < T_D$

dumbbell (see Figure 4) and a singularity must occur for this surface. (To continue

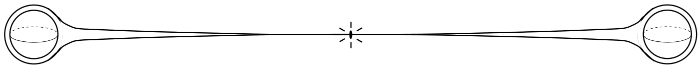

FIGURE 4. A singularity occurs

the flow past such a singularity see [4] and [5], or [6]–[9].)

If the initial surface is convex, the situation is better: Huisken proved 1984, that the solution stays convex and shrinks in finite time to a round point. This means, that if one rescales the solution suitably (e.g., by keeping the area constant), it converges in finite time uniformly to a round sphere.

In $\mathbb{R}^2$, the situation is even better: if one starts the (mean) curvature flow (also called curve shortening flow in this case) with an embedded closed curve, the solution stays embedded and converges in finite time to a round point. This is a result of Grayson (see [11]).

Nonetheless, the curve shortening flow can develop singularities also in the plane, if one starts with a curve that is not initially embedded. The example in Figure 5 is due to Angenent (see [2]): here, the inner loop suffers from its higher curvature and therefore shrinks faster than the outer loop. A singularity forms in finite time. By rescaling the solution suitably, e.g., by keeping the maximal curvature constant, the rescaled solution converges to a very particular limit, namely the curve $x = -\log\cos y$ (see Figure 6): Angenent has shown, that the blow-up of every so called type II