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SELF-SIMILAR EXPANDING SOLUTIONS FOR THE PLANAR NETWORK FLOW

by

Rafe Mazzeo & Mariel Saez

Abstract. — We prove the existence of self-similar expanding solutions of the curvature flow on planar networks where the initial configuration is any number of half-lines meeting at the origin. This generalizes recent work by Schnürer and Schulze which treats the case of three half-lines. There are multiple solutions, and these are parametrized by combinatorial objects, namely Steiner trees with respect to a complete negatively curved metric on the unit ball which span k specified points on the boundary at infinity. We also provide a sharp formulation of the regularity of these solutions at $t = 0$.

Résumé **(Solutions dilatantes auto-similaires pour le flot des réseaux planaires)**
Nous montrons l'existence de solutions autosimilaires dilatantes du flot de la courbure sur des réseaux planaires où la configuration initiale est un nombre quelconque de demi-droites qui se rencontrent à l'origine. Celà généralise un travail récent de Schnürer et Schulze qui ont traité le cas de trois demi-droites. Ce sont des solutions multiples et elles sont paramétrées par des objets combinatoires qui sont les arbres de Steiner de la boule unité munie d'une métrique à coubure négative. Ils engendrent k points spécifiés sur le bord à l'infini. Nous donnons aussi une formulation optimale de la régularité à l'instant $t = 0$.

1. Introduction

The detailed analysis of the curve-shortening flow for embedded closed curves in the plane was an early success in the field of geometric flows. It is not hard to extend this theory to include curves with boundary which are either fixed (a Dirichlet condition) or constrained to lie on the boundary of a convex domain, for example (a Neumann condition).

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Slightly more generally, one might consider the flow by curvature for networks of curves.

Definition 1.1. — *A planar network is a finite union of embedded arcs and properly embedded half-lines* $\{\gamma_i\}$ *such that for each* $i \neq j$, $\gamma_i \cap \gamma_j$ *is either empty or else consists of one (or both) boundary points of each curve. Each boundary points of every* γ_i *is either a boundary or interior vertex. These intersections are called the (interior) vertices of the network. The boundary of the network consists of the set of points which are endpoints of exactly one of these curves. The number of curves intersecting at each interior vertex is called the valence of that vertex. We always assume that all curves are at least* $\mathcal{C}^2$

One must formulate the evolution equation for the flow near vertices weakly, and this is commonly done using Brakke's ideas [4]. However, certain features of the short-time existence, and many other aspects of the long-time existence and convergence, for this network flow have proved elusive. The first thorough analytic attack on this problem was undertaken by Mantegazza, Novaga and Tortorelli [7] several years ago. Their results are primarily directed at networks which consist of only three arcs meeting at equal angles at a single interior vertex. The primary difficulties include the choice of boundary conditions at this vertex, and later (for more complicated initial networks) the possibility of creation and annihilation of such vertices during the flow. They obtain some interesting convergence results under certain hypotheses, but many difficult questions remain.

Let us say that a network is *regular* if each interior vertex is trivalent and the curves meet at equal angles (of $2\pi/3$) there. Brakke flow preserves regularity of vertices, at least locally in t. More precisely, if the edges of a network are evolving independently of one another by curvature flow in such a way that at later times t, the edges still fit together into a network and all vertices are regular, then this network is evolving by Brakke flow. Slightly more generally, if a network is evolving so that at each vertex, the sum of the unit tangents to all incoming curves vanishes, then it too is evolving by Brakke flow. In this paper we shall restrict to 'classical' solutions of Brakke flow satisfying the first condition, i.e. that all vertices are regular.

It is important to move beyond the restricted class of initial configurations considered in [7] and consider the flow starting at a more general network with multivalent vertices. One motivation is that such 'nonregular' networks may appear at discrete values of time when vertices collide, so it is important to understand how to flow past them. This paper takes the first step in proving short-time existence for the curvature flow on networks when the initial configuration is one of these more general networks. We prove here the existence of self-similar solutions, i.e. expanding solitons for the flow, when the initial network is a finite union of half-lines intersecting at the origin. The solution with this initial condition is far from unique, but we are able to describe the set of all solutions which are regular when $t > 0$. Finally, we also present a somewhat new perspective which leads to a sharp regularity statement at 'irregular vertices' at time 0. In a forthcoming sequel to this paper we shall apply our results here to prove short-time existence for the network flow starting from fairly general initial networks.

Our approach is inspired by a quite recent paper by Schnürer and Schulze [10], in which they prove the existence and uniqueness of a self-similar solution when the initial condition consists of three half-lines meeting at the origin, but not necessarily in equal angles. Their solution is regular for $t > 0$ and remains a union of three properly embedded arcs meeting at a common vertex. They do not state the precise trajectory of this vertex. In contrast, the self-similar solutions here, which start from a union of at least four half-lines meeting at 0, immediately break up into a (possibly disconnected) regular network with multiple interior vertices and remain so for all later times. As we explain, the trajectories of these vertices are easy to determine. This 'explosion' of a nonregular vertex into a more complicated network provides the model for the short-time existence for the general network flow, and the multiplicity of self-similar solutions corresponds to the nonuniqueness of solutions with a given initial condition.

Imposing self-similarity is tantamount to a dimension reduction of the equation, which transforms this problem into an ODE. Schnürer and Schulze derive certain convexity properties of solutions of this ODE, which were key to their analysis. However, there is a somewhat broader and more natural geometric picture which we explain here, that solutions of this ODE are geodesics for a certain complete metric on the plane, and the curvature properties of this metric provide a concise explanation for those convexity properties. Furthermore, the identification of these curves with geodesics allows us to use variational arguments to prove the existence of the more complicated regular networks which provide the solutions to our problem.

To state our main result, let us introduce some notation. Let B be the ball, regarded as the stereographic compactification of $\mathbb{R}^2$. A union of k half-lines C_0 meeting at the origin in $\mathbb{R}^2$ determines a finite collection of points $p_1, \dots, p_k \in \partial B$. Define the metric

$$g = e^{x^2+y^2}(dx^2 + dy^2).$$

This is complete and negatively curved, with curvature tending to 0 at infinity. We can now state our main result.

Main Theorem. — *Let C_0 be a finite union of half-lines in the plane meeting at 0, and $p_1, \dots, p_k$ the corresponding points on ∂B. The set of self-similar solutions of the curve-shortening flow with initial condition C_0 is in bijective correspondence with the* *set of possibly disconnected regular networks on B, each arc of which is a geodesic for g, with boundary the k prescribed points at infinity. There always exists at least one (and often very many) connected geodesic Steiner tree with these asymptotic boundary values, and indeed, there also exist geodesic Steiner trees with precisely s components corresponding to any 'increasing' partition $\{1,\dots,k\} = I_1 \cup \dots, I_s$, where each I_j consists of a consecutive string of integers, with each $|I_j| > 1$. Finally, these self-similar solutions lift to a smooth family of networks on the parabolic blowup of $\mathbb{R}^2 \times \mathbb{R}^+$ at $x = y = t = 0$.*

This general picture was understood qualitatively by Brakke, and hinted at in an appendix to his book [4]. Unbeknownst to us when we were doing this work, many of the specific facts presented here were discovered in slightly different forms by Tom Ilmanen and Brian White in the mid '90's. Some discussion of this appears in [3] and [6], but the focus in those papers is mostly on the higher dimensional case. We hope that this independent and more elementary discussion of the one-dimensional case, along the lines of [10], is not unwelcome. There are some interesting new points too, including the enumeration of self-similar treelike expanding solutions in terms of (nonelementary) combinatorial data, i.e. the number of Steiner trees in $(\mathbb{R}^2, g)$ spanning the k given points at infinity, as well as the formulation of regularity. The first author wishes to thank Brian White for several very helpful conversations, and in particular for explaining certain aspects of the Brakke flow, and more importantly, 'size minimization' in the class of flat chains mod k, which provides a shortcut to the existence result of §3 which circumvents a more explicit but longer synthetic approach. Frank Morgan also provided some useful comments. The second author wishes to thank Marilyn Daily and Felix Schulze for pointing out appropriate references. R.M. was supported by the NSF under grant DMS-050579.

The next section describes solutions of the dimension reduced equation and the geometry of the metric g; the main existence result for regular networks with prescribed asymptotes is proved in §3; finally, we make some remarks about regularity at $t = 0$ in §4, explaining the last assertion of the main theorem.

2. Self-similar solutions of curve-shortening flow

Let γ_0 be an immersed curve in the plane. The curve-shortening flow with initial condition γ_0 is the evolution leading to the family of curves γ_t, $t \geq 0$, where

$$\frac{d}{dt}\gamma_t = \kappa(\gamma_t)\nu(\gamma_t);$$

here κ is the curvature and ν the unit normal to γ_t. The well-known theorem of Grayson asserts that if γ_0 is closed and embedded, then γ_t remains embedded and