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ON THE KÄHLER CLASSES OF CONSTANT SCALAR CURVATURE METRICS ON BLOW UPS

by

Claudio Arezzo & Frank Pacard

Abstract. — We consider when the blow up at a finite number of points of a compact Kähler constant scalar curvature manifold has constant scalar curvature. The particular case when the manifold does not admit any Hamiltonian holomorphic vector fields is discussed.

Résumé **(Sur les classes de Kähler des métriques à courbure scalaire constante définies sur des éclatements)**

Nous considérons quand l'éclatement en un nombre fini de points d'une variété compacte Kählérienne à courbure scalaire constante a une courbure scalaire constante. Le cas particulier où la variété n'admet aucun champ de vecteur holomorphe Hamiltonien est considéré.

1. Introduction

In this short paper we address the following question:

Problem 1.1. — *Given a compact constant scalar curvature Kähler manifold* (M, J, g, ω)*, of complex dimension* $m := dim_{\mathbb{C}} M$*, and having defined*

$$\triangle := \{(p_1, \ldots, p_n) \in M^n \quad : \quad \exists\, a \neq b \quad p_a = p_b\},$$

characterize the set $\mathcal{PW} = \{(p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n)\} \subset (M^n \setminus \triangle) \times (0, +\infty)^n$ *for which* $\tilde{M} = \mathrm{Bl}_{p_1,\ldots,p_n} M$*, the blow up of* M *at* $p_1, \ldots, p_n$ *has a constant scalar curvature Kähler metric (cscK from now on) in the Kähler class*

$$\pi^*[\omega] - (\alpha_1\, PD[E_1] + \cdots + \alpha_n\, PD[E_n]),$$

where the $PD[E_j]$ *are the Poincaré duals of the* $(2m-2)$*-homology classes of the exceptional divisors of the blow up at* p_j*.*

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This general problem is too complicated and its solution is likely to pass through the solution of well known conjectures relating the existence of cscK metrics with the K-stability of the polarized manifold.

Yet, more specific questions are treatable and could give light also on these ambitious programs. The first natural narrowing of Problem 1.1 is to require that not just one Kähler class has a cscK representative, but that this is the case for a whole segment in the Kähler cone of $\tilde{M}$ touching the boundary at a point of the form $\pi^*[\omega]$, where ω is (necessarily) a cscK form on M. Analytically this amounts to the following:

Problem 1.2. — *Given a compact Kähler constant scalar curvature manifold* (M, J, g, ω) *characterize the set* $\mathcal{APW} = \{(p_1, \dots, p_n, a_1, \dots, a_n)\} \subset (M^n \setminus \triangle) \times (0, +\infty)^n$ *such that* $\tilde{M} = \mathrm{Bl}_{p_1,\dots,p_n} M$ *has a constant scalar curvature Kähler metric in the class*

$$\pi^*[\omega] - \varepsilon^2 \left(a_1 \, PD[E_1] + \cdots + a_n \, PD[E_n]\right),$$

for all ε *sufficiently small. Here* $\mathcal{APW}$ *refers to "asymptotic points and weights", namely points and weights in this singular perturbation setting.*

Hence we can consider $(\alpha_1, \dots, \alpha_n)$ as an asymptotic direction in the Kähler cone for which canonical representative can be found. It is immediate to extract from [2] the following :

Theorem 1.1. — *Assume that* (M, J, g, ω) *is a constant scalar curvature compact Kähler manifold without any nontrivial hamiltonian holomorphic vector field. Then* $\mathcal{APW} = (M^n \setminus \Delta) \times (0, +\infty)^n$.

The presence of hamiltonian holomorphic vector fields greatly enhances the difficulty and the interest of the problem. In [1] the authors have attacked this problem and found an interplay between its solution and the behavior of the hamiltonian holomorphic vector fields at the p_j that we briefly recall.

First recall that the Matsushima-Lichnerowicz Theorem asserts that the space of hamiltonian holomorphic vector fields on (M, J, ω) is also the complexification of the real vector space of holomorphic vector fields Ξ which can be written as

$$\Xi = X - i \, J \, X,$$

where X is a Killing vector field which vanish somewhere on M. Let us denote by $\mathfrak{h}$, the space of hamiltonian holomorphic vector field and by

$$\xi_\omega : M \longmapsto \mathfrak{h}^*$$

the *moment* map which is defined by requiring that if $\Xi \in \mathfrak{h}$, the function $\zeta_\omega := \langle \xi_\omega, \Xi \rangle$ is a (complex valued) Hamiltonian for the vector field Ξ, namely the unique solution of

$$-\bar{\partial} \zeta_\omega = \tfrac{1}{2} \, \omega(\Xi, -),$$

which is normalized by

$$\int_M \zeta_\omega \, dvol_g = 0.$$

With these notations, the result we have obtained in [1] reads:

Theorem 1.2. — *Assume that (M, J, g, ω) is a constant scalar curvature compact Kähler manifold and that $p_1, \ldots, p_n \in M$ and $a_1, \ldots, a_n > 0$ are chosen so that:*

(i) $\xi_\omega(p_1), \ldots, \xi_\omega(p_n)$ *span* $\mathfrak{h}^*$
(ii) $\sum_{j=1}^n a_j^{m-1}\, \xi_\omega(p_j) = 0 \in \mathfrak{h}^*$.

Then, there exist $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there exists on $\tilde{M} = \mathrm{Bl}_{p_1,\ldots,p_n} M$, a constant scalar curvature Kähler metric g_ε associated to the Kähler form

$$\omega_\varepsilon \in \pi^*[\omega] - \varepsilon^2 \,(a_{1,\varepsilon}\, PD[E_1] + \cdots + a_{n,\varepsilon}\, PD[E_n]),$$

where

$$|a_{j,\varepsilon} - a_j| \leq c\, \varepsilon^{\frac{2}{2m+1}}. \tag{1}$$

Finally, the sequence of metrics $(g_\varepsilon)_\varepsilon$ converges to g in $\mathcal{C}^\infty(M \setminus \{p_1, \ldots, p_n\})$.

Therefore, in the presence of nontrivial hamiltonian holomorphic vector fields, the number of points which can be blown up, their position, as well as the possible Kähler classes on the blown up manifold have to satisfy some constraints.

It is not hard to see from the proof in [1] that the mapping

$$(a_1, \ldots, a_n) \longmapsto (a_{1,\varepsilon}, \ldots, a_{n,\varepsilon})$$

is continuous. Indeed, this follows from the construction itself which only uses fixed point theorems for contraction mappings and hence the metric we obtain depends smoothly on the parameters of the construction.

Theorem 1.2 has two major drawbacks : First, we lose control on the Kähler classes on $\tilde{M}$ for which constant scalar curvature Kähler metrics can be constructed, second there are severe restrictions on the set of points and asymptotic directions.

The key idea to fill these gaps is to note that the construction of [1] is in fact a construction of the Riemannian metric g_ε and this is reflected by the fact that the sequence of metrics constructed converges to the initial metric g and also in the fact that condition (ii) really depends on the choice of the metric g.

Now, on the one hand, the origin of (ii) stems from the existence of hamiltonian holomorphic vector fields on (M, J) and in fact (ii) imposes on the choice of the asymptotic directions $(a_1, \ldots, a_n)$ as many constraints as the dimension of $\mathfrak{h}$.

On the other hand, the existence of hamiltonian holomorphic vector fields is also related to the non-uniqueness of the constant scalar curvature Kähler metric on M. More precisely, $\mathfrak{h}$ is the Lie algebra of the group of automorphisms of (M, J, g, ω) and as such also parameterizes near g the space of constant scalar curvature Kähler metrics in a given Kähler class $[\omega]$ and for a given scalar curvature. Observe that this space has dimension equal to $\dim \mathfrak{h}$. Therefore, we can Apply the result of Theorem 1.2 not only to the metric g itself but also to the pull back of g by any biholomorphic transformation.

Since condition (ii) depends on the choice of the metric, if we are only interested in the Kähler classes on the blown up manifold, we get more flexibility in the choice of the asymptotic parameters (observe that the dimension of the space of constant scalar curvature Kähler metrics near g (with fixed scalar curvature) is precisely equal to the number of constraints on the choice of the asymptotic parameters). This observation allows us to complement the result of Theorem 1.2 and get the:

Theorem 1.3. — *Assume that* (M, J, g, ω) *is a constant scalar curvature compact Kähler manifold and that* $p_1, \dots, p_n \in M$ *and* $a_1, \dots, a_n > 0$ *are chosen so that:*

(i) $\xi_\omega(p_1), \dots, \xi_\omega(p_n)$ *span* $\mathfrak{h}^*$ *(genericity condition);*
(ii) $\sum_{j=1}^n a_j^{m-1}\, \xi_\omega(p_j) = 0 \in \mathfrak{h}^*$ *(balancing condition);*
(iii) *no element of* $\mathfrak{h}$ *vanishes at every point* $p_1, \dots, p_n$ *(general position condition).*

Then $(p_1, \dots, p_n, a_1, \dots, a_n) \in \mathcal{APW}$.

Therefore, we can indeed prescribe the exact value of the asymptotic direction in which the Kähler classes in perturbed at the expense of imposing that no hamiltonian holomorphic vector field vanishes at every point we blow up.

The genericity condition is purely technical and it does not seem to hide any deep geometric nature. Indeed, as observed in [1]:

Lemma 1.1. — *With the above notations, assume that* $n \geq \dim \mathfrak{h}$. *Then, the set of points* $(p_1, \dots, p_n) \in M^n \setminus \triangle$ *satisfying the genericity condition is open and dense.*

The balancing condition is certainly the heart of the problem, encoding the relevant stability property of $\tilde{M}$. For example when all the a_j are rationals, the balancing condition is easily translated in the Chow polystability of the cycle $\sum_j a_j^{m-1} p_j$ with respect to the action of the automorphism group of M.

In a remarkable recent paper Stoppa [13] has proved, among other things, that it if the cycle $\sum_j a_j^{m-1} p_j$ is Chow unstable, then $(p_1, \dots, p_n, a_1, \dots, a_n)$ does not lie in $\mathcal{APW}$. With a beautifully careful algebraic analysis he has in fact related a destabilizing configuration for the points to a destabilizing configuration of the blown up manifold giving a quantitative measure of the reciprocal unstabilities.

Going back to our problem, we first observe that the combination of the three above condition still leaves flexibility in the choices:

Theorem 1.4. — *With the above notations, assume that* $n \geq \dim \mathfrak{h} + 1$ *then, the set of points* $((p_1, \dots, p_n), (a_1, \dots, a_n)) \in (M^n \setminus \triangle) \times (0, \infty)^n$ *such that condition (i), (ii) and (iii) are fulfilled is open in* $(M^n \setminus \triangle) \times (0, \infty)^n$.

Openness in the choice of the points was already contained in [1]. What we will prove in this short pPer is the openness in the choice of the asymptotic directions.