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FULLY NONLINEAR EQUATIONS, ELLIPTICITY, AND CURVATURE PINCHING

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Abstract. — In this article we describe some second-order PDEs from conformal geometry, and the connections between the ellipticity of these equations and various notions of curvature pinching. Particular study is made of the PDEs which result from prescribing symmetric functions of the eigenvalues of the Schouten tensor.

***Résumé* (Les équations non-linéaires, ellpiticité et le pincement de la courbure).** — Dans cet article, nous décrivons quelques EDP du second ordre issus de la géométrie conforme, et les liens entre l'ellipticité de ces équations et les diverses notions de courbure pincée. Nous étudions en particulier l'EDP qui résulte en prescrivant les fonctions symétriques des valeurs propres du tenseur de Schouten.

1. Introduction

In this article I want to describe some second-order PDEs from conformal geometry, and the interesting connection between the ellipticity of these equations and various notions of curvature pinching. The reader should be forewarned that I am selectively choosing results in a broad and rapidly advancing field and make no pretense to providing a general survey, which can be found, for example, in [45]. Rather, I wish to emphasize those results in field that exploit the fully nonlinear structure of the equations for geometric ends.

To begin, suppose (M^n, g) is a compact Riemannian manifold of dimension $n \geq 3$ without boundary. Let $\mathrm{Riem} = \mathrm{Riem}(g)$, $\mathrm{Ric} = \mathrm{Ric}(g)$, and $R = R(g)$ denote respectively the Riemannian curvature tensor, the Ricci curvature tensor, and the scalar curvature of g. From the perspective of conformal geometry the Ricci tensor and scalar curvature are not the most natural components of the decomposition of

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Riem. To see why, let $W = W(g)$ denote the Weyl curvature tensor, and let $A = A(g)$ denote the *Schouten tensor*, defined by

$$A = \frac{1}{(n-2)}\left(\mathrm{Ric} - \frac{1}{2(n-1)}Rg\right). \tag{1.1}$$

Then one can decompose the curvature tensor as

$$\mathrm{Riem} = W + A \wedge g, \tag{1.2}$$

where $\wedge$ is the exterior product, extended in the natural way to the bundle of symmetric $(0,2)$-tensors. It is well known that the Weyl tensor if conformally invariant: e.g., if $\widehat{g} = e^{-2u}g$ is a conformal metric, then

$$W(\widehat{g}) = e^{-2u}W(g). \tag{1.3}$$

From this fact and (1.2) we conclude that under a conformal change of metric the behavior of the curvature tensor is completely determined by the behavior of the the Schouten tensor.

What is the behavior of the Schouten tensor? If $\widehat{g} = e^{-2u}g$, then

$$A(\widehat{g}) = A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g, \tag{1.4}$$

where all covariant derivatives are with taken with respect to the background metric g. Incidentally, this formula also explains why we write our conformal factors as e^{-2u}, since then the conformal Schouten tensor is of the form

$$A(\widehat{g}) = \nabla^2 u + \cdots, \tag{1.5}$$

where $\cdots$ denotes lower order (in derivatives) terms. Roughly speaking, we are interested in the PDEs which result from prescribing symmetric functions of the eigenvalues of the Schouten tensor. The first such equations were introduced in the thesis of Jeff Viaclovsky [**42**], where he considered the elementary symmetric polynomials.

To make this more precise, for $1 \leq k \leq n$, let $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ denote the elementary symmetric polynomial of degree k:

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \cdots i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1.6}$$

Let $f \in C^\infty$ be a given smooth function; we would like to find a conformal metric $\widehat{g} = e^{-2u}g$ with

$$\sigma_k(A(\widehat{g})) = \psi(x), \tag{1.7}$$

where $\sigma_k(A(\widehat{g}))$ denotes σ_k applied to the eigenvalues of $A(\widehat{g})$. By (1.4), equation (1.7) is equivalent to the following PDE in the conformal factor u:

$$\sigma_k\left(A + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g\right) = \psi(x)e^{-2ku}. \tag{1.8}$$

The exponential weight on the right-hand side results from the convention that the eigenvalues of the bilinear form on the left-hand side are computed with respect to the background metric g.

When $k = 1$, $\sigma_1(A)$ is the trace of the Schouten tensor, which by (1.1) is just a multiple of the scalar curvature:

$$\sigma_1(A) = \frac{1}{2(n-1)} R. \tag{1.9}$$

Therefore, (1.7) is the equation of prescribed scalar curvature. In this case one typically writes $\widehat{g} = v^{4/(n-2)} g$ and in place of (1.8) we have

$$-\frac{4(n-1)}{(n-2)} \Delta v + R(g) v = R(\widehat{g}) v^{\frac{(n+2)}{(n-2)}}. \tag{1.10}$$

Unlike the scalar curvature equation, (1.8) is not in general variational; or at least not in an obvious way. Let

$$[g] = \{\widehat{g} = e^{-2u} g \mid u \in C^\infty\} \tag{1.11}$$

denote the conformal class of g, and

$$[g]_1 = \{\widehat{g} \in [g] \mid \mathrm{Vol}(\widehat{g}) = 1\} \tag{1.12}$$

denote conformal metrics of unit volume. Consider the functionals $\mathcal{F}_2 : [g]_1 \to \mathbb{R}$ defined by

$$\mathcal{F}_k[g] = \int \sigma_k(A)\, dV. \tag{1.13}$$

Up to a constant $\mathcal{F}_1$ is just the total scalar curvature; a metric g is a critical point of

$$\mathcal{F}_1\big|_{[g]_1}$$

if and only if g has constant curvature. There is of course an analogous variational formulation of conformal metrics with variable curvature. For other values of k, however, the picture is more complicated:

Theorem 1.1. — (See [42]) *Suppose $k \neq n/2$.*

(i) *If $k = 1$ or 2, then a metric g is critical for $\mathcal{F}_k \mid_{[g]_1}$ if and only if $\sigma_k(A(g)) =$* const.

(ii) *Suppose $k \geq 3$ and (M^n, g) is locally conformally flat. Then g is critical for $\mathcal{F}_k \mid_{[g]_1}$ if and only if $\sigma_k(A(g)) =$* const.

Before moving on the some more technical aspects required for the analysis of (1.8), we make some comments.

Remark 1. — The condition $k \neq n/2$ is due to the conformal invariance of $\mathcal{F}_{n/2}$ in certain cases. For example, when $k = 2$ and $n = 4$, by the Chern-Gauss-Bonnet theorem we have

$$2\pi^2 \chi(M^4) = \frac{1}{4} \int \|W\|^2\, dV + \int \sigma_2(A)\, dV. \tag{1.14}$$

The pointwise formula (1.3) implies that the L^2-integral of the Weyl curvature is conformally invariant, and it follows that the integral of $\mathcal{F}_2$ is conformally invariant in dimension 4.

When $n = 2k \geq 6$ and (M^n, g) is locally conformally flat, then

$$\chi(M^n) = c_n \int \sigma_{n/2}(A)\, dV,$$

so that $\mathcal{F}_{n/2}$ is conformally invariant; see [42].

Remark 2. — Viaclovsky's Theorem shows the distinctiveness of the functionals $\mathcal{F}_1$ and $\mathcal{F}_2$. This extends beyond their variational properties within a fixed conformal class; see [23].

Remark 3. — There are other equations involving symmetric functions of the eigenvalues of A which are the Euler equations of variational integrals. In [11], Ge-Lin-Wang studied the functional

$$\mathcal{F}_{2,1}[g] = \frac{\displaystyle\int \sigma_2(A)\, dV}{\left(\displaystyle\int \sigma_1(A)\, dV\right)^{\frac{n-4}{n-2}}}. \tag{1.15}$$

In $n \neq 4$, then critical points of $\mathcal{F}_{2,1}$ satisfy

$$\frac{\sigma_2(A)}{\sigma_1(A)} = \text{const.} \tag{1.16}$$

Remark 4. — Expanding on the previous example, in general one can consider equations of the form

$$F[A(\widehat{g})] = \psi(x), \tag{1.17}$$

where $F : \mathcal{R}^n \to \mathbb{R}$ is a symmetric function of n variables satisfying certain *structural* conditions. We will give some specific examples below.

In addition to variational properties, a crucial difference between the scalar curvature equation and equation (1.7) is the fact that the latter is in general not elliptic. Therefore, in the next section we will provide a characterization of ellipticity.

2. Ellipticity

Consider a general second order differential equation defined on a domain $\Omega \subset \mathcal{R}^n$ which we write as

$$F[x, u, \nabla u, \nabla^2 u] = 0. \tag{2.1}$$

Here, $F : \mathcal{U} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$. To simplify the exposition, let us suppose that $F : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ and the corresponding equation is of the form

$$F[x, \nabla^2 u] = 0. \tag{2.2}$$